A NOTE ON THE CONNECTEDNESS PROBLEM FOR NEST ALGEBRAS

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Abstract. It has been conjectured that a certain operator \( T \) belonging to the group \( \mathcal{G} \) of invertible elements of the algebra \( \text{Alg} \mathbb{Z} \) of doubly infinite upper-triangular bounded matrices lies outside the connected component of the identity in \( \mathcal{G} \). In this note we show that \( T \) actually lies inside the connected component of the identity of \( \mathcal{G} \).

Let \( T \) be the unit circle in the complex plane with normalized Lebesgue measure. For \( 1 < p < \infty \), let \( H^p \) be the usual Hardy space of all functions in \( L^p(T) \) that have analytic extensions to the open unit disk \( D \). Let \( \mathcal{H} = L^2(T) \) and let \( \mathcal{B}(\mathcal{H}) \) be set of all bounded linear operators on \( \mathcal{H} \). Let \( W \in \mathcal{B}(\mathcal{H}) \) be the shift operator: \( (Wf)(e^{i\theta}) = e^{i\theta} f(e^{i\theta}) \). In this paper, we consider the nest \( \{ W^n H^2 : n \in \mathbb{Z} \} \) of subspaces of \( L^2(T) \), and its associated nest algebra, 

\[
\text{Alg} \mathbb{Z} = \{ T \in \mathcal{B}(\mathcal{H}) : TW^n H^2 \subseteq W^n H^2 \text{ for all } n \in \mathbb{Z} \}.
\]

A question which has been unanswered for several years is the following:

Question. Is the group of invertible elements of the Banach algebra \( \text{Alg} \mathbb{Z} \) connected in the norm topology?

It is frequently conjectured that the answer to this question is no. The reason for conjecturing a negative answer is because of a strong analogy between nest algebras and analytic function theory. We refer the reader to the book by Davidson [1] for details and more background on this question.

For each \( f \in L^\infty(T) \), let \( M_f \in \mathcal{B}(\mathcal{H}) \) be the multiplication operator,

\[
M_f \phi = f \phi, \quad \phi \in L^2(T).
\]

Note that for \( f \in H^\infty \), we have \( M_f \in \text{Alg} \mathbb{Z} \). Let \( a \) be a positive real number and set

\[
h(z) = \frac{ai}{\pi} \log \left( \frac{1 + z}{1 - z} \right).
\]
Then \( h \) is a conformal map of the open unit disk onto the unbounded vertical strip \( \{ z \in \mathbb{C} : -a < \text{Re}(z) < a \} \).

If \( f = \exp(h) \) then it is easy to see that both \( f \) and \( 1/f \) are \( H^\infty \) functions and moreover, that \( f \) is not the exponential of any \( H^\infty \) function. Therefore \( f \) cannot be connected to the constant function 1 via a norm continuous path within the group of invertible elements of the Banach algebra \( H^\infty \). For this reason, the operator \( M_f \) has been suggested as a possible example of an operator which cannot be connected to the identity via a norm continuous path inside the group of invertibles in \( \text{Alg} \mathbb{Z} \).

The purpose of this note is to show that in fact, \( M_f \) may be connected to the identity via a norm continuous path of invertible elements in \( \text{Alg} \mathbb{Z} \).

Before giving the proof we pause for some terminology and to make a few simple remarks.

Let \( \mathcal{A} \) be a unital Banach algebra with unit \( I \). Say that an invertible element \( a \) of \( \mathcal{A} \) may be connected to the identity if there exists a norm continuous function \( f : [0, 1] \rightarrow \mathcal{A} \) such that \( f(0) = a \), \( f(1) = I \), and \( f(t) \) is an invertible element of \( \mathcal{A} \) for each \( t \). The algebra \( \mathcal{A} \) has the connectedness property if every invertible element of \( \mathcal{A} \) may be connected to the identity. We use the term symmetry to describe a square root of the identity in a unital Banach algebra \( \mathcal{A} \). Such elements have spectrum contained in the set \( \{-1, 1\} \) and hence are connected to the identity. In fact, if \( \gamma(t) \) is an arc in the complex plane connecting \(-1\) to \(1\) which does not pass through the origin, then

\[
   \sigma(t) = \frac{I + \gamma(t)}{2} + \gamma(t) \frac{I - \gamma(t)}{2}
\]

is a norm continuous path of invertible elements of \( \mathcal{A} \) which connects the symmetry \( \gamma \) to the identity \( I \).

The algebra

\[
   \mathcal{D} = \text{Alg} \mathbb{Z} \cap (\text{Alg} \mathbb{Z})^*
\]

is a von Neumann subalgebra of \( \text{Alg} \mathbb{Z} \) and since any von Neumann algebra has the connectedness property, we see that any invertible operator in \( \mathcal{D} \) can be connected to the identity in \( \text{Alg} \mathbb{Z} \).

**Remark.** Let \( \alpha \) be a complex number of unit modulus and let \( g \in L^\infty(T) \). Let

\[
   g_\alpha(z) = g(\alpha z), \quad z \in T,
\]

and define a unitary operator \( S_\alpha \in \mathcal{D} \) by

\[
   S_\alpha e_n = \alpha^n e_n,
\]

where \( e_n(e^{i\theta}) = e^{in\theta} \) is the usual orthonormal basis for \( L^2(T) \).

We then have

\[
   S_\alpha M_g S_\alpha^* = M_{g_\alpha}.
\]

Note that by the above remarks, \( M_g \) and \( M_{g_\alpha} \) belong to the same connectedness class of invertibles in \( \text{Alg} \mathbb{Z} \).

We now show that \( M_f \) can be connected to the identity. Note that \( h(z) = -h(-z) \). It follows that we have

\[
   f(z)f(-z) = 1 \quad \text{for all } z \in \overline{\mathcal{D}}.
\]
If $S = S_{-1}$, equation (2) yields,

$$SM_fSM_f = I.$$ 

Hence both $S$ and $SM_f$ are symmetries in $\text{Alg} \mathbb{Z}$ and

$$M_f = S(SM_f).$$

Therefore $M_f$ can be connected to the identity in $\text{Alg} \mathbb{Z}$. Moreover, equation (1) enables one to obtain an explicit path connecting $M_f$ to the identity.

**Question.** Let $m$ be a conformal mapping of the disk onto itself and set $g = f \circ m$. Is $M_g$ connected to the identity in $\text{Alg} \mathbb{Z}$? Note that the remark above shows that if $m$ is a rotation, then this is the case.

**Remark.** Let $R$ be any proper open subset of the complex plane that is simply connected and satisfies $R = -R$. Then $0 \in R$ and if $h$ is any conformal map from the disk onto $R$ with $h(0) = 0$, we have $h(z) = -h(-z)$. (Indeed, the function $g(z) = -h(-z)$ is also a conformal map of the disk onto $R$. Since $h(0) = g(0)$ and $h'(0) = g'(0)$, the Riemann mapping theorem implies $g = h$.) The argument given above now shows that if we assume that $\{\Re(z) : z \in R\}$ is bounded and set $f = \exp(h)$, then $M_f$ is a product of two symmetries in $\text{Alg} \mathbb{Z}$ and hence is connected to the identity in $\text{Alg} \mathbb{Z}$.

**REFERENCES**
