EXTREMAL COMPRESSIONS OF CLOSED OPERATORS

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Abstract. Let $X$ be a Banach space, $A$ a closed linear operator on $X$, and $\lambda_1, \ldots, \lambda_n$ isolated eigenvalues of $A$ of finite multiplicity. If $P$ is a projection on $X$ such that $\lambda_1, \ldots, \lambda_n$ belong to the resolvent of the compression of $A$ on the range of $P$ it is easy to see that

$$\dim N(P) \geq \max \{\dim N(\lambda_i - A) : 1 \leq i \leq n\}.$$

It is shown that there exist such projections where we have equality in this inequality.

Let $X$ be a complex Banach space. For an operator $A$ in $X$ (not necessarily everywhere defined) we denote by $D(A)$, $N(A)$, and $R(A)$ the domain, the kernel, and the range of $A$, respectively. If $P$ is a bounded linear projection on $X$ with $P(D(A)) \subseteq D(A)$ the compression of $A$ on $R(P)$ is the linear operator $A_P$ defined in the closed subspace $R(P)$ by $D(A_P) = D(A) \cap R(P)$ and $A_P y = P A y$ for $y \in R(P)$. It is easy to see that $A_P$ is injective (i.e., $N(A_P) = \{0\}$) iff $N(A) \cap R(P) = \{0\}$ and $R(A_P) \cap N(P) = \{0\}$. Therefore $\dim N(P) \geq \dim N(A)$ if $A_P$ is injective.

The collection of projections $P$ on $X$ with $P(D(A)) \subseteq D(A)$ will be denoted by $P_A(X)$. The main result of this note is the following:

**Theorem.** Let $A$ be a closed linear operator in a Banach space $X$ and let $\lambda_1, \ldots, \lambda_n$ be isolated eigenvalues of $A$ of finite (algebraic) multiplicity. Let $\Omega$ be a subset of $\mathbb{C}$ such that $\Omega \neq \mathbb{C}$ and $\Omega \cap \sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. If $\zeta \in \mathbb{C} \setminus \Omega$ then there is a projection $P \in P_A(X)$ such that $\Omega \subseteq \rho(A_P)$, $\sigma(A_P) = \sigma(A) \setminus \Omega \cup \{\zeta\}$ and

$$\dim N(P) = \max_{\lambda \in \Omega} \dim N(\lambda - A).$$

In [1, 2] Islamov proved a similar result for finite-dimensional perturbations of $A$. It is well known that many results on compact or finite-dimensional perturbations have analogues in terms of compressions to subspaces of finite codimension, see [5, 6, 7]. The theorem above is another example of this relationship.

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The proof will be divided into two steps. The main step is the following lemma which is the theorem in the finite-dimensional case. For its proof we need some facts about $\lambda$-matrices (for details see [4]).

A $\lambda$-matrix is a matrix over the ring of polynomials in $\lambda$ over $\mathbb{C}$. Let $B$ be an $n \times n$ matrix over $\mathbb{C}$ and $L_B(\lambda) := \lambda I - B$. For $l = 1, \ldots, n$ define the polynomial $d_l(\lambda)$ to be the greatest common divisor (gcd) of all minors of $L_B(\lambda)$ of order $l$ and $d_0(\lambda) \equiv 1$.

$$i_l(\lambda) := d_l(\lambda)/d_{l-1}(\lambda) \quad (l = 1, \ldots, n)$$

are called the invariant polynomials of $L_B(\lambda)$. Following holds (see [4, 4.10]): Two $n \times n$ matrices $A, B$ are similar iff $L_A$, $L_B$ have the same invariant polynomials. Two $\lambda$-matrices $L_1(\lambda), L_2(\lambda)$ are said to be equivalent if there are $\lambda$-matrices $P(\lambda), Q(\lambda)$ with nonzero constant determinants such that $L_1(\lambda) = P(\lambda)L_2(\lambda)Q(\lambda)$.

**Lemma.** Let $B$ be an $n \times n$ matrix and $\zeta \in \mathbb{C}$. There is a subspace $N$ of $\mathbb{C}^n$ and a projection $P$ onto $N$ such that for the compression $B_P$ holds $\sigma(B_P) = \{\zeta\}$ and

$$\dim N(P) = \max_{\lambda \in \mathbb{C}} \dim N(\lambda - B).$$

**Proof.** Let $i_{r+1}(\lambda), \ldots, i_n(\lambda)$ be the nonconstant invariant polynomials of $L_B(\lambda)$. At first we show

$$(*) \quad n - r = \max \{\dim N(\lambda - B) : \lambda \in \mathbb{C}\}.$$ 

It is well known (see e.g. [4, p. 143, 148]) that $i_l(\lambda)$ divides $i_{l'}(\lambda)$ for $l \leq l'$. Thus for $\lambda_0 \in \mathbb{C}$ we get the implications $i_l(\lambda_0) = 0 \Rightarrow i_k(\lambda_0) = 0$ for $l \leq k \leq n$ and $i_l(\lambda_0) \neq 0 \Rightarrow i_k(\lambda_0) \neq 0$ for $1 \leq k \leq l$. By [4, Theorem 4.9.1] $L_B(\lambda)$ is equivalent to $\text{diag}(i_1(\lambda), \ldots, i_n(\lambda))$. So for every $\lambda_0 \in \mathbb{C}$ we have $\dim N(\lambda_0 - B) = \dim N(\text{diag}(i_1(\lambda_0), \ldots, i_n(\lambda_0))) = n - \min\{l : i_l(\lambda_0) = 0\} + 1$ (with $\min\{\} = n + 1$ if $i_n(\lambda_0) \neq 0$). Taking the maximum on each side, we get $(*)$.

Now consider a polynomial $p(\lambda) = \lambda^l + \alpha_{l-1}\lambda^{l-1} + \cdots + \alpha_0$ ($l > 0$) and its companion matrix

$$L(p) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -\alpha_0 & \cdots & -\alpha_{l-2} & -\alpha_{l-1} \end{pmatrix}.$$

Considering invariant polynomials we next show that it is possible to choose suitable $\beta_0, \ldots, \beta_{l-1}$ such that $L(p)$ and

$$C_\zeta(p) = \begin{pmatrix} \zeta & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \zeta & 1 \\ \beta_0 & \cdots & \beta_{l-2} & \beta_{l-1} \end{pmatrix}$$

are similar.
By \([4, 4.11]\) \(\lambda - L(p)\) has the invariant polynomials \(1, \ldots, 1, p\). \(\lambda - C_\xi(p)\) has constant nonzero minors of order \(1, \ldots, l - 1\) independently of the choice of \(\beta_0, \ldots, \beta_{l-1}\). So the first \(l - 1\) invariant polynomials are identically 1. The last one is

\[
\det[\lambda - C_\xi(p)] = (\lambda - \zeta)^l - (\beta_{l-1} - \zeta)(\lambda - \zeta)^{l-1} - \beta_{l-2}(\lambda - \zeta)^{l-2} - \cdots - \beta_0
\]

which induces the choice of \(\beta_0, \ldots, \beta_{l-1}\) depending on \(p\).

\(B\) is similar to \(\text{diag}(L(i_{r+1}), \ldots, L(i_n))\) and so similar to \(D := \text{diag}(C_\xi(i_{r+1}), \ldots, C_\xi(i_n))\). Let \(U\) be an invertible \(n \times n\) matrix with \(B = UDU^{-1}\) and \(l_j = \deg i_j(\lambda)\). Define \(N_1\) to be the subspace of \(C^n\) spanned by the canonical vectors \(e_j\) for \(j \notin \{i_{r+1}, i_{r+1} + i_{r+2}, \ldots, i_{r+1} + i_{r+2} + \cdots + i_n\}\).

We set \(N := UN_1\) and \(P = UP_1U^{-1}\), where \(P_1\) denotes the orthogonal projection onto \(N_1\). Then \(P\) is a projection onto \(N\), \(\dim N(P) = \dim N(P_1) = n - r\). \(B_P = VDP_1V^{-1}\) where \(V := U|_{N_1}: N_1 \to N\). \(B_P\) and \(D_P\) have the same eigenvalues. \(D_P\) is the matrix \(D\) where rows and columns with indices \(i_{r+1}, i_{r+1} + i_{r+2}, \ldots, i_{r+1} + i_{r+2} + \cdots + i_n\) are deleted, so \(\sigma(B_P) = \{\zeta\}\). The lemma is proved.

**Proof of the theorem.** \(\Omega \cap \sigma(A) = \{\lambda_1, \ldots, \lambda_n\}\) is a bounded part of \(\sigma(A)\), separated from \(\sigma(A)\setminus\{\lambda_1, \ldots, \lambda_n\}\). By [3, Theorem 6.17] we have a decomposition of \(A\) according to a decomposition \(X = M_1 \oplus M_2\) in such a way that we have for \(A_{P_1}, A_{P_2}(P_1\) the projection onto \(M_1\) along \(M_2\), \(P_2 = I - P_1\):

\[
\dim P_1X = \dim M_1 < \infty, \quad D(A_{P_1}) = M_1
\]

\[
\sigma(A_{P_1}) = \{\lambda_1, \ldots, \lambda_n\}, \quad \sigma(A_{P_2}) = \sigma(A)\setminus\{\lambda_1, \ldots, \lambda_n\},
\]

and

\[
\dim N(\lambda - A) = \dim N(\lambda - A_{P_1}) \quad \text{for} \quad \lambda \in \Omega.
\]

By the preceding lemma there is a subspace \(N\) of \(M_1\) and a projection \(\tilde{P}\) of \(M_1\) onto \(N\) such that \(\dim M_1/N = \max\{\dim N(\lambda - A_{P_1}) : \lambda \in \mathbb{C}\}\) and \(\sigma((A_{P_1})_P) = \{\zeta\}\).

Set \(M := N \oplus M_2\) and \(P := \tilde{PP}_1 + P_2\). Then we have

\[
\text{codim}_X M = \text{codim}_{M_1} N = \max_{\lambda \in \mathbb{C}} \dim N(\lambda - A_{P_1}) = \max_{\lambda \in \Omega} \dim N(\lambda - A),
\]

\[
P^2 = P, \quad R(P) = M \quad \text{and} \quad \sigma(A_P) = \sigma(A)\setminus\Omega \cup \{\zeta\}.
\]

The proof is complete.

With this result it is possible to show a corollary for the Browder spectrum analogously to [7].

**Corollary.** Let \(A\) be a bounded operator on a complex Banach space \(X\) and \(\varepsilon > 0\). Then there is an extremal compression \(A_P\) of \(A\) in such a way that \(\sigma(A_P)\) is contained in the \(\varepsilon\)-neighborhood \(U\) of the Browder spectrum \(\sigma_b(A)\) of \(A\) and

\[
\dim N(P) = \max_{\lambda \notin U} \dim N(\lambda - A).
\]

**Proof.** Consider \(\sigma(A)\setminus U\), with \(U = \{\lambda \in \mathbb{C} : \text{dist}(\lambda, \sigma_b(A)) < \varepsilon\}\). This is a finite set because \(\sigma(A)\setminus U\) is a compact set of isolated points. Applying the theorem to \(A\), \(\Omega := \sigma(A)\setminus U\) and an arbitrary \(\zeta \in \sigma_{\text{ess}}(A)\) shows the corollary.
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