CENTRALIZERS OF EXPANDING MAPS ON THE CIRCLE

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Abstract. We prove here that the elements of an open and dense subset of expanding maps on the circle have trivial centralizers; i.e., the maps commute only with their own powers. Using a theorem proved in [1], we deduce that the result is also true for an open and dense subset of immersions of $S^1$.

1. Introduction

Let $\text{Imm}^n(S^1)$ be the space of $C^n$ immersions of the circle $S^1$ (i.e., mappings $f$ for which $f'(x) \neq 0$) endowed with the $C^n$ topology. An immersion $f: S^1 \to S^1$ is expanding iff $|f'(x)| > 1$ for all $x \in S^1$. Let $\text{Exp}^n(S^1)$ denote the set of $C^n$ expanding maps of $S^1$.

We continue here to study, initiated in [1], of centralizers of immersions of the circle. Recall that for $f \in \text{Imm}^n(S^1)$, its centralizer $Z(f)$ in $\text{Imm}^n(S^1)$ is defined as the set of elements that commute with $f$. We say that $f$ has trivial centralizer if $Z(f)$ is reduced to the iterates $\{f^n, n \in \mathbb{N}\}$ of $f$.

The purpose of this paper is to prove the following result.

Theorem. For an open and dense subset of $\text{Exp}^n(S^1)$ ($n > 1$), the centralizer is trivial.

It follows from [1] that a similar result is true for an open and dense subset of $\text{Imm}^\infty(S^1) - \text{Exp}^\infty(S^1)$. Hence we have proved

Corollary. There is an open and dense subset of $\text{Imm}^\infty(S^1)$ whose elements have trivial centralizers.

This result is an extension to immersions of a theorem of Kopell [2, Theorem 3], who showed the triviality of the centralizer for an open and dense subset of diffeomorphisms of the circle.

A fundamental tool for the proof of the theorem is [1, Lemma 2.1] which is an extension to expanding maps on the circle of a result of Kopell [2, Theorem 6].

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2. Proof of the theorem

We begin by recalling some basic concepts and establishing preliminary results.

Formally, we will think of the circle as $\mathbb{R}/\mathbb{Z}$ and use $\pi$ to denote the canonical projection. Thus every continuous map $f$ of the circle has countably many lifts, i.e., continuous maps $F: S^1 \to S^1$ satisfying

$$f \circ \pi = \pi \circ F.$$ 

Any two such lifts differ by an integer, and the unique integer $d$ satisfying

$$F(x + 1) = F(x) + d$$

for all lifts $F$ and all $x$ is called the degree of $f$.

Let $f_d$ denote the immersion of $S^1$ given by $f_d(z) = zd$.

**Lemma 2.1.** Let $f$ be an expanding map of the circle, and suppose that $f'(x) \neq f'(y)$ for all different fixed points of $f$. Then

(a) If $g \in Z(f)$, then $g$ fixes the fixed points of $f$.
(b) Let $h: S^1 \to S^1$ be a homeomorphism of the circle such that $h \circ f = f_n \circ h$, where $n = \text{degree } f$. If $g \in Z(f)$, then $h \circ g = f_m \circ h$, where $m = \text{degree } g$.

**Proof.** (a) Let $x$ be a fixed point of $f$. Then

$$f \circ g(x) = g \circ f(x) = g(x),$$

and

$$f'(g(x)) = f'(x).$$

These properties and the hypothesis imply that $g(x) = x$.

(b) We have that

$$f_n \circ (h \circ g \circ h^{-1}) = h \circ f \circ g \circ h^{-1} = h \circ g \circ f \circ h^{-1} = (h \circ g \circ h^{-1}) \circ f_n.$$ 

Hence, by [1, Lemma 2.1], $h \circ g \circ h^{-1} = \omega f_m$, where $\omega$ is an $(n - 1)$th root of unity and $m = \text{degree } g$. By (a), $h^{-1}(1)$ is a fixed point of $g$. Thus

$$1 = h \circ g \circ h^{-1}(1) = \omega f_m(1) = \omega.$$ 

Therefore, $h \circ g = f_m \circ h$, and the lemma is proved.

We will use the following simple fact.

**Lemma 2.2.** Let $f: S^1 \to S^1$ be a $C^n$ endomorphism of the circle, and let $\alpha: S^1 \to S^1$ be a $C^n$ diffeomorphism. If $Z(\alpha^{-1} \circ f \circ \alpha)$ is trivial, then $Z(f)$ is trivial.

**Proof.** Let $g \in Z(f)$. Then $\alpha^{-1} \circ g \circ \alpha \in Z(\alpha^{-1} \circ f \circ \alpha)$. Hence, by hypothesis, $\alpha^{-1} \circ g \circ \alpha = \alpha^{-1} \circ f^k \circ \alpha$ for some $k \in \mathbb{N}$. Thus, $g = f^k$ and so $Z(f)$ is trivial.

We are now ready to prove the theorem. Let

$$\mathcal{U} = \{f \in \text{Exp}^n(S^1) : f'(x) \neq f'(y) \text{ for different fixed points } x, y \text{ of } f\}.$$
This set is clearly an open and dense subset of $\text{Exp}^n(S^1)$. Hence, it suffices to show that the elements of $\mathbb{Z}$ have trivial centralizers. Let $f \in \mathbb{Z}$. By a result of Shub [3], there exists an order-preserving homeomorphism $h : S^1 \to S^1$ such that

$$h \circ f = f_n \circ h,$$

where $n = \text{degree } f$.

By using a rotation of $S^1$, if necessary, we may assume by Lemma 2.2 that $f(1) = h(1) = 1$. Let $g \in Z(f)$. By Lemma 2.1(a), $g(1) = 1$. Hence, if $F$, $G$, $H$ are the lifts of $f$, $g$, and $h$, respectively, then

$$F(0) = G(0) = H(0) = 0,$$

$$H \circ F(x) = nH(x) \quad \text{for all } x \in \mathbb{R},$$

and by Lemma 2.1(b), $H \circ G(x) = mH(x)$ where $m = \text{degree } g$. Let $l \in \mathbb{N}$ such that $n^l \leq m \leq n^{l+1}$. Then

$$m = n^l + p \quad \text{for some } 0 \leq p \leq n^l(n - 1).$$

We claim that $G'(0) = (F'(0))^l$. In fact, let $t_k \in \mathbb{R}$ such that $H(t_k) = 1/n^k$. Let $\varepsilon > 0$ be given. Since $H$ is an order-preserving homeomorphism, we can choose $k \in \mathbb{N}$ such that

$$0 \leq H^{-1}\left(\frac{1}{n^k} + \frac{p}{n^{k+1}}\right) - t_k \leq \varepsilon.$$

Since $F^{-1}$ is a contraction, we have that

$$(*) \quad \lim_{j \to \infty} F^{-j}\left(H^{-1}\left(\frac{1}{n^k} + \frac{p}{n^{k+1}}\right)\right) = 0$$

and

$$0 \leq F^{-j}\left(H^{-1}\left(\frac{1}{n^k} + \frac{p}{n^{k+1}}\right)\right) - F^{-j}(t_k) \leq \varepsilon$$

for all $j \in \mathbb{N}$. Thus

$$(**) \quad 1 \leq \frac{F^{-j}(H^{-1}(\frac{1}{n^k} + \frac{p}{n^{k+1}}))}{F^{-j}(t_k)} \leq 1 + \varepsilon \quad \text{for all } j \in \mathbb{N}.$$  

Let $y_j = F^{-j}(t_k)$. Then

$$\lim_{j \to \infty} y_j = 0$$

and

$$\frac{1}{n^k} = H(t_k) = H \circ F^j(y_j) = n^jH(y_j).$$

Hence, $H(y_j) = 1/n^{k+j}$ for all $j \in \mathbb{N}$. These properties, together with the fact that $F^{-1} \circ H^{-1}(nx) = H^{-1}(x)$ for all $x \in \mathbb{R}$, imply that

$$G'(0) = \lim_{j \to \infty} \frac{G(y_j) - G(0)}{y_j} = \lim_{j \to \infty} \frac{H^{-1}(mH(y_j))}{y_j}$$

$$= \lim_{j \to \infty} \frac{H^{-1}(\frac{m}{n^{k+j}})}{y_j} = \lim_{j \to \infty} \frac{F^{-j-l} \circ H^{-1}(\frac{mn^{l-j}}{n^{k+j}})}{y_j}$$

$$= \lim_{j \to \infty} \frac{F^l(F^{-j}(H^{-1}(\frac{1}{n^k} + \frac{p}{n^{k+1}})))}{y_j} \frac{F^{-j}(H^{-1}(\frac{1}{n^k} + \frac{p}{n^{k+1}}))}{F^{-j}(t_k)}.$$

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But, by \((*)\),
\[
\lim_{j \to \infty} \frac{F^j(F^{-j}(H^{-1}(\frac{x}{n^t} + \frac{p}{n^{t+1}}))))}{F^{-j}(H^{-1}(\frac{x}{n^t} + \frac{p}{n^{t+1}})))} = (F^j)'(0).
\]
So, by \((**)\),
\[
(F^j)'(0) \leq G'(0) \leq (F^j)'(0) + \varepsilon.
\]
Therefore \(G'(0) = (F^j)'(0)\), and the claim is proved.

By Sternberg [41, there exists a \(C^\omega\)-diffeomorphism \(\alpha: \mathbb{R} \to \mathbb{R}\) such that \(\alpha(0) = 0\) and \(\alpha \circ F \circ \alpha^{-1}(x) = F'(0)x\) for all sufficiently small \(x\). Let \(L = \alpha \circ F \circ \alpha^{-1}\). Then \(\alpha \circ G \circ \alpha^{-1}\) commutes with \(L\). Thus, for all sufficiently small \(x\), we have that
\[
(\alpha \circ G \circ \alpha^{-1})(x)F'(0) = (\alpha \circ G \circ \alpha^{-1}) \left( L \left( \frac{x}{F'(0)} \right) \right) L' \left( \frac{x}{F'(0)} \right)
\]
\[
= L' \left( \alpha \circ G \circ \alpha^{-1} \left( \frac{x}{F'(0)} \right) \right) (\alpha \circ G \circ \alpha^{-1})' \left( \frac{x}{F'(0)} \right)
\]
\[
= F'(0)(\alpha \circ G \circ \alpha^{-1})' \left( \frac{x}{F'(0)} \right),
\]
so
\[
(\alpha \circ G \circ \alpha^{-1})'(x) = (\alpha \circ G \circ \alpha^{-1})' \left( \frac{x}{F'(0)} \right).
\]
This and the fact that \(|F'(0)| > 1\) imply that
\[
(\alpha \circ G \circ \alpha^{-1})'(x) = (\alpha \circ G \circ \alpha^{-1})' \left( \frac{x}{(F'(0))^{q}} \right) \text{ for all } q \in \mathbb{N}.
\]
Since \((\alpha \circ G \circ \alpha^{-1})'\) is continuous, we have that
\[
(\alpha \circ G \circ \alpha^{-1})'(x) = (\alpha \circ G \circ \alpha^{-1})'(0) = G'(0).
\]
This and the claim above imply that, for all sufficiently small \(x\),
\[
\alpha \circ G \circ \alpha^{-1}(x) = G'(0)x = (F'(0))^{j}x = L^{j}(x),
\]
so \(G(x) = F^{j}(x)\) for all sufficiently small \(x\). It follows from this that there exists an open interval \(I\) in \(S^{1}\) such that \(g(x) = f^{q}(x)\) for all \(x \in I\). Since \(f\) is expanding, for all \(y \in S^{1}\) there exist \(x \in I\) and \(q \in \mathbb{N}\) such that \(f^{q}(x) = y\). Thus
\[
g(y) = g \circ f^{q}(x) = f^{q} \circ g(x) = f^{q} \circ f^{j}(x) = f^{j}(y).
\]
Therefore \(g = f^{j}\), and the theorem is proved.

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**References**


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