SOME GENERAL LOCAL VARIATIONAL PRINCIPLES

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ABSTRACT. Local variational min-sup characterizations are presented for the real spectrum of a selfadjoint operator pencil. Instead of minimizing over all subspaces of fixed codimension as in the classical result, the new characterizations minimize over subspaces that are close to extremal subspaces. In this way, the entire real spectrum, including continuous spectrum, can be characterized.

1. INTRODUCTION

For a selfadjoint operator that is bounded below, there is a classical minimax theorem that characterizes the discrete end of the spectrum. Over the years, this minimax characterization has played a very important role in the investigation of spectra of selfadjoint operators. It is natural to ask whether similar characterizations exist for the entire spectrum of a selfadjoint operator, or more generally for the real spectrum of a selfadjoint operator pencil $\lambda A - B$. In recent works [AM, NY], a minimax theorem of the classical form was established for certain eigenvalues of a selfadjoint matrix (or operator) pencil. In this paper it is our intention to consider a minimax characterization that has a weaker form but applies to all the real spectrum.

We first note that the minimax principle is closely related to the stationary property of the Rayleigh quotient, which is a local result. It turns out that this is an appropriate framework for obtaining a general minimax theorem. We present a local minimax characterization for the real spectrum of a selfadjoint operator pencil. One extremum in our theorem is taken over subspaces that are close to an extremal subspace instead of over all subspaces of specific codimension as in the classical case. It is in this sense that we regard it as a local characterization. In the rest of this section, we present the basic theorem and some direct implications. The proofs are in §3, and a more complicated application is given in §4.

Let H be a separable Hilbert space. We first introduce a neighborhood system in the set of closed subspaces. Given such a subspace S_0, we choose an orthonormal basis $\mathscr{B}$ for it, and we consider subspaces spanned by elements

that are slightly perturbed from $\mathscr{B}$. Specifically, we introduce the following definition.

Definition 1.1. Let S_0 and T_0 be two closed subspaces such that $H = S_0 \dotplus T_0$. Then for $\varepsilon > 0$,

$$N_\varepsilon(S_0, T_0) = \left\{ \operatorname{span}\{e_n + e_n'\}_{n=0}^{+\infty} : e_n' = \sum_{\substack{m=-\infty}}^{+\infty} \varepsilon_{mn} e_m, \ \sum_{\substack{m=-\infty \\ n=0}}^{+\infty} |\varepsilon_{mn}|^2 < \varepsilon \right\}$$

is called an ε-neighborhood of S_0 relative to T_0, where $\{e_n\}_{n=0}^{+\infty}$ and $\{e_n\}_{-n=1}^{+\infty}$ are orthonormal bases of S_0 and T_0 respectively.

We show in §2 that the definition of $N_\varepsilon(S_0, T_0)$ is independent of the choices of orthonormal bases $\{e_n\}_{n=0}^{\infty}$ and $\{e_n\}_{-n=1}^{+\infty}$ and thus pertains only to the subspaces S_0 and T_0.

Let A, B be two bounded selfadjoint operators on H, and let

$$(1.1) \qquad\qquad C_\pm = \{x \in H : \pm(Ax, x) > 0\}.$$

Our starting point is the following basic pair of assumptions for some fixed $\lambda \in \mathbb{R}$.

(1) H admits the direct sum decomposition $H = S_\lambda \dotplus T_\lambda$ with S_λ, T_λ closed and

$$\lambda = \sup_{x \in S_\lambda \cap C_+} (Bx, x)/(Ax, x).$$

(2) There is a fixed $\delta > 0$, so that for any $\gamma > 0$, there exist a subspace $T_\gamma \supset T_\lambda$ and an element $x_\gamma \in T_\gamma \cap S_\lambda$ satisfying $(Ax_\gamma, x_\gamma) > \delta\|x_\gamma\|^2$ and

$$\inf_{x \in T_\gamma \cap C_+} (Bx, x)/(Ax, x) \geq \lambda - \gamma.$$

We now state the local minimax theorem.

Theorem 1.2. *Let A, B be two bounded selfadjoint operators on a separable Hilbert space H. If assumptions (1) and (2) hold for some $\lambda \in \mathbb{R}$, then there is $\varepsilon > 0$ such that*

$$\lambda = \min_{S \in N_\varepsilon(S_\lambda, T_\lambda)} \sup_{x \in S \cap C_+} (Bx, x)/(Ax, x).$$

To apply this abstract theorem to concrete problems, we need to construct S_λ, T_λ, and T_γ so that assumptions (1) and (2) are satisfied. Here we present two direct applications. If E is the resolution of the identity for a selfadjoint operator C on H, so $Cx = \int_{-\infty}^{\infty} \lambda \, dE(\lambda)x$ for all $x \in H$, then we write

$$(1.2) \qquad\qquad E_\lambda = E(]-\infty, \lambda]).$$

Theorem 1.3. *Let E be the resolution of the identity for a bounded selfadjoint operator B on H. If*

$$(1.3) \qquad\qquad E(]\lambda_0 - \gamma, \lambda_0]) \neq 0 \quad \text{for all } \gamma > 0$$

(so λ_0 is an eigenvalue of B or a point of continuous spectrum not on the left boundary), then there is $\varepsilon > 0$, such that

$$\lambda_0 = \min_{S \in N_\varepsilon(S_0, T_0)} \sup_{x \in S, \, x \neq 0} (Bx, x)/(x, x),$$

where $S_0 = \Re(E_{\lambda_0})$, $T_0 = \Re(I - E_{\lambda_0})$ *(see* (1.2)).

We say that λ_0 is an eigenvalue of the operator pencil $P: \lambda \to \lambda A - B$ if $Bx = \lambda_0 Ax$ for some nonzero $x \in H$, i.e. 0 is an eigenvalue of $\lambda_0 A - B$. Also λ_0 is A-positive (resp. A-negative) if $x \in C_+$ (resp. $x \in C_-$) of (1.1).

Theorem 1.4. *If λ_0 is an A-positive eigenvalue of P, let E be the resolution of the identity for $\lambda_0 A - B$ and let $S_0 = \Re(E_0)$, $T_0 = \Re(I - E_0)$. Then there is $\varepsilon > 0$ such that*

$$\lambda_0 = \min_{S \in N_\varepsilon(S_0, T_0)} \sup_{x \in S \cap C_+} (Bx, x)/(Ax, x).$$

Our final application is Theorem 4.2, which extends Theorem 1.3 to Pontryagin space and Theorem 1.4 to include continuous spectrum in an appropriate sense. Both extensions are, however, at the cost of extra conditions on the pair (A, B).

All the results in this paper have been restricted to min-sup characterizations and A-positive eigenvalues. There are dual characterizations of max-inf type, and also of A-negative eigenvalues, which can be obtained similarly and are left to the reader.

2. Preliminaries

In this section, we prove an essential lemma on intersection of subspaces. First, however, we clarify Definition 1.1.

Theorem 2.1. *The definition of $N_\varepsilon(S_0, T_0)$ is independent of the choice of orthonormal bases for S_0 and T_0.*

Proof. If $S_0 = \text{span}\{f_n\}_{n=0}^{+\infty}$, $T_0 = \text{span}\{f_n\}_{-n=1}^{+\infty}$, where $\{f_n\}_{n=0}^{+\infty}$ and $\{f_n\}_{-n=1}^{+\infty}$ are orthonormal respectively, let

$$e_n = \sum_{l=0}^{+\infty} \alpha_{nl} f_l \quad \text{for } n \geq 0, \qquad e_n = \sum_{-l=1}^{+\infty} \alpha_{nl} f_l \quad \text{for } n \leq -1$$

and

$$f_l = \sum_{n=0}^{+\infty} \beta_{ln} e_n \quad \text{for } l \geq 0, \qquad f_l = \sum_{-n=1}^{+\infty} \beta_{ln} e_n \quad \text{for } l \leq -1.$$

Then

$$(2.1) \qquad \sum_{l=0}^{+\infty} \beta_{nl} \alpha_{lm} = \sum_{l=0}^{+\infty} \alpha_{nl} \beta_{lm} = \delta_{nm} \quad \text{for } m, n \geq 0,$$

$$(2.2) \qquad \sum_{-l=1}^{+\infty} \beta_{ml} \alpha_{ln} = \sum_{-l=1}^{+\infty} \alpha_{nl} \beta_{lm} = \delta_{nm} \quad \text{for } m, n \leq -1,$$

and

$$(2.3) \qquad \sum_{n=0}^{+\infty} \beta_{ln} \bar{\beta}_{kn} = \delta_{lk} \quad \text{for } l, k \geq 0.$$

Now, for a subspace $S = \text{span}\{e_n + e'_n\}_{n=0}^{+\infty}$ with $e'_n = \sum_{m=-\infty}^{+\infty} \varepsilon_{mn} e_m$ and $\sum_{m=-\infty}^{+\infty},\, n=0 |\varepsilon_{mn}|^2 < \varepsilon$, by (2.1) we have

$$S = \text{span}\left\{\sum_{n=0}^{+\infty} \beta_{ln}(e_n + e'_n)\right\}_{l=0}^{+\infty} = \text{span}\left\{f_l + \sum_{k=-\infty}^{+\infty} \varepsilon'_{kl} f_k\right\}_{l=0}^{+\infty},$$

where

$$\varepsilon'_{kl} = \sum_{n=0}^{+\infty} \beta_{ln} \sum_{-m=1}^{+\infty} \varepsilon_{mn}\alpha_{mk} \quad \text{for } k \le -1 \text{ and } l \ge 0$$

and

$$\varepsilon'_{kl} = \sum_{n=0}^{+\infty} \beta_{ln} \sum_{m=0}^{+\infty} \varepsilon_{mn}\alpha_{mk} \quad \text{for } k \ge 0 \text{ and } l \ge 0.$$

Furthermore, by (2.2) and (2.3), we have $\alpha_{jk} = \bar{\beta}_{kj}$. Then after a computation, we reach

$$\sum_{\substack{k=-\infty \\ l=0}}^{+\infty} |\varepsilon'_{kl}|^2 = \sum_{\substack{m=-\infty \\ n=0}}^{\infty} |\varepsilon_{mn}|^2.$$

The proof is complete. □

A basic problem in the construction of minimax principles is whether two given subspaces intersect nontrivially. In the classical case, this intersection can be deduced from dimension arguments. In the general case, both subspaces in question could be of infinite dimension. We show that given two nontrivially intersecting subspaces, they still intersect if one of the subspaces is perturbed slightly in the sense of Definition 1.1.

Lemma 2.2. *Let A be a bounded selfadjoint operator on a separable Hilbert space $H = S_0 \dotplus T_0$ with S_0, T_0 closed. Given $\delta > 0$, there is $\varepsilon > 0$, such that if $S \in N_\varepsilon(S_0, T_0)$, $T_0 \subset T$, and $x_T \in S_0 \cap T$ with $(Ax_T, x_T) > \delta\|x_T\|^2$, then there is $x \in S \cap T$ with $(Ax, x) > 0$.*

Proof. Let $S_0 = \text{span}\{e_n\}_{n=0}^{\infty}$, $T_0 = \text{span}\{e_n\}_{-n=1}^{+\infty}$, and $S = \text{span}\{e_n + e'_n\}_{n=0}^{+\infty}$ with $e'_n = \sum_{m=-\infty}^{+\infty} \varepsilon_{mn} e_m$ and $\sum_{m=-\infty}^{+\infty} {}_{n=0} |\varepsilon_{mn}|^2 < \varepsilon < 1$. For $x_T \in S_0 \cap T$, we can write $x_T = \sum_{n=0}^{+\infty} \beta_n e_n$. We then consider the following equation in α_n:

$$(2.4) \qquad\qquad \alpha_n + \sum_{m=0}^{+\infty} \alpha_m \varepsilon_{nm} = \beta_n,$$

for $n = 0, 1, 2, \dots$. Equivalently, this can be written in infinite matrix form

$$(2.5) \qquad\qquad (I + E)\alpha = \beta$$

on l_2, where

$$E = \begin{bmatrix} \varepsilon_{00} & \varepsilon_{01} & \varepsilon_{02} & \cdots \\ \varepsilon_{10} & \varepsilon_{11} & \varepsilon_{12} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}, \quad a = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \end{bmatrix}, \quad \text{and} \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \end{bmatrix}.$$

Since $\|E\|_2 \leq \sum_{m=0,\ n=0}^{+\infty} |\varepsilon_{mn}|^2 < \varepsilon < 1$, (2.5) has the unique solution $\alpha = (I+E)^{-1}\beta$. Moreover, $\|\alpha\|_2 \leq \|\beta\|_2/(1 - \|E\|_2) \leq \|x_T\|_2/(1-\varepsilon)$. We now let

$$x = \sum_{n=0}^{+\infty} \alpha_n(e_n + e'_n) \in S.$$

Then by (2.4), we have $x = y_0 + x_T$, where

$$y_0 = \sum_{-m=1}^{+\infty} \left(\sum_{n=0}^{+\infty} \varepsilon_{mn}\alpha_n \right) e_m \in T_0 \subset T,$$

so $x \in S \cap T$. Furthermore

$$\|y_0\|^2 = \sum_{-m=1}^{+\infty} \left| \sum_{n=0}^{+\infty} \varepsilon_{mn}\alpha_n \right|^2 \leq \left(\sum_{-m=1}^{+\infty} \sum_{n=0}^{+\infty} |\varepsilon_{mn}|^2 \right) \left(\sum_{n=0}^{+\infty} |\alpha_n|^2 \right) < \frac{\varepsilon\|x_T\|^2}{(1-\varepsilon)^2}$$

and

$$\begin{aligned}
(Ax, x) &= (Ax_T, x_T) + 2\operatorname{Re}(Ax_T, y_0) + (Ay_0, y_0) \\
&> \delta\|x_T\|^2 - 2\|x_T\| \cdot \|A\| \cdot \|y_0\| - \|A\| \, \|y_0\|^2 \\
&\geq \left(\delta - \frac{2\sqrt{\varepsilon}}{1-\varepsilon}\|A\| - \frac{\varepsilon}{(1-\varepsilon)^2}\|A\| \right) \cdot \|x_T\|^2.
\end{aligned}$$

Thus, choosing ε small enough to ensure $\delta - (2\sqrt{\varepsilon}/(1-\varepsilon) + \varepsilon/(1-\varepsilon)^2)\|A\| > 0$, we have $(x, Ax) > 0$. This completes the proof. $\square$

3. Proofs of the main theorems

Proof of Theorem 1.2. Choose δ as in assumption (2). Thus for any $\gamma > 0$, there is a subspace $T_\gamma \supset T_\lambda$ and $x_\gamma \in T_\gamma \cap S_\lambda$ such that $(Ax_\gamma, x_\gamma) > \delta\|x_\gamma\|^2$ and

$$\inf_{x \in T_\gamma \cap C_+} (Bx, x)/(Ax, x) \geq \lambda - \gamma.$$

Now let ε correspond to δ in Lemma 2.2. Then for given $S \in N_\varepsilon(S_\lambda, T_\lambda)$, there is $x_0 \in S \cap T_\gamma \cap C_+$. Thus

$$\sup_{x \in S \cap C_+} \frac{(Bx, x)}{(Ax, x)} \geq \frac{(Bx_0, x_0)}{(Ax_0, x_0)} \geq \inf_{x \in T_\gamma \cap C_+} \frac{(Bx, x)}{(Ax, x)} \geq \lambda - \gamma.$$

Noting that γ is any positive number, we have

$$\sup_{x \in S \cap C_+} (Bx, x)/(Ax, x) \geq \lambda.$$

This together with assumption (1) proves the theorem. $\square$

As pointed out in §1, several applications can be derived from Theorem 1.2. We now proceed to the proofs of the remaining theorems.

Proof of Theorem 1.3. It is enough to check assumptions (1) and (2) with $A = I$. For $x \in S_0 = \Re(E_{\lambda_0})$

$$(Bx, x) = \int_{-\infty}^{\lambda_0} \lambda(dE(\lambda)x, x) \leq \lambda_0(x, x).$$

In view of (1.3),

$$(3.1) \qquad \sup_{x \in S_0, \; x \neq 0} (Bx, x)/(x, x) = \lambda_0$$

and we have verified assumption (1). Now, for any $\gamma > 0$, let $T_\gamma = \Re(I - E_{\lambda_0 - \gamma})$. Then $T_\gamma \supset T_0$. Furthermore, $S_0 \cap T_\gamma = \Re(E_{\lambda_0} - E_{\lambda_0 - \gamma}) \neq \{0\}$ so x_γ exists with $\delta = 1/2$, and

$$\inf_{x \in T_\gamma, \; x \neq 0} \frac{(Bx, x)}{(x, x)} \geq \lambda_0 - \gamma.$$

Hence, assumption (2) is also satisfied.   □

Proof of Theorem 1.4. Again, we only need to check assumptions (1) and (2). Let x_0 be an eigenvector corresponding to λ_0 with $(Ax_0, x_0) > 0$. For $x \in S_0 = \Re(E_0)$,

$$((B - \lambda_0 A)x, x) \leq 0,$$

i.e.

$$(Bx, x) \leq \lambda_0 (Ax, x).$$

Since $x_0 \in S_0 \cap C_+$, we have

$$\sup_{x \in S_0 \cap C_+} (Bx, x)/(Ax, x) = \lambda_0.$$

Now, for any $\gamma > 0$, let $T_\gamma = \Re(I - E(]-\infty, 0[))$. Then $T_\gamma \supset T_0$ and $x_0 \in S_0 \cap T_\gamma$. Obviously $(Ax_0, x_0) > \delta(x_0, x_0)$ with $\delta = (Ax_0, x_0)/2(x_0, x_0) > 0$. Also, for $x \in T_\gamma$,

$$((B - \lambda_0 A)x, x) \geq 0.$$

Thus

$$\inf_{x \in T_\gamma \cap C_+} (Bx, x)/(Ax, x) \geq \lambda_0 > \lambda_0 - \gamma.$$

Hence both assumptions (1) and (2) are satisfied.   □

4. A VARIATIONAL PRINCIPLE IN PONTRYAGIN SPACE

The basis for this section is a direct sum of selfadjoint pencils $P_1 \oplus P_2$ on $H = H_1 \oplus H_2$ where $\dim H_1$ is finite, Theorem 1.4 applies to P_1 and Theorem 1.3 applies to P_2. Specifically, we write

$$(4.1) \qquad P_1(\lambda) = \lambda A_1 - B_1, \qquad P_2(\lambda) = \lambda I_2 - B_2,$$

where A_1 is nonsingular. We remark that a variety of pencils on a given Hilbert space H can be transformed into such direct sums [B]. Evidently, the operator

$$(4.2) \qquad Q = A^{-1}(B_1 \oplus B_2)$$

is selfadjoint in the Pontryagin space H_A defined as H but with the inner product generated by $A = A_1 \oplus I_2$. (Note that the Hilbert majorant topology of H_A is just that of H.) Conversely, any selfadjoint operator L on a Pontryagin space admits such a direct sum decomposition, provided a certain nondegeneracy condition holds [B, p. 337] or equivalently provided there are no singular critical points [L, p. 40]. As a simple example, if L is compact then the above conditions hold if $N(L)$ is nondegenerate (e.g. if L is $1-1$) (cf. [BS, Theorem 1]).

We have the following result for P_1.

Lemma 4.1. *For any* $\lambda \in \mathbb{R}$, *there exist complementary subspaces* S_λ^1 *and* T_λ^1 *of* H_1, *such that*

$$\sup_{x \in S_\lambda^1 \cap C_+} \frac{(B_1 x, x)}{(A_1 x, x)} \leq \lambda \leq \inf_{x \in T_\lambda^1 \cap C_+} \frac{(B_1 x, x)}{(A_1 x, x)}.$$

Furthermore, if λ *is an* A-*positive eigenvalue of* P_1 *with a corresponding eigenvector* $x_0 \in C_+$, *then* $x_0 \in S_\lambda^1$.

A construction of S_λ^1, T_λ^1 for certain eigenvalues λ is contained in the proof of Theorem 3.1 of [NY] in the context of matrix pencils, but in fact this construction applies to any real number and not just the eigenvalues. The details are omitted here, but we remark that these subspaces S_λ^1 and T_λ^1 involve splitting spectral subspaces for $A_1^{-1} B_1$ in a fashion that is in general not unique.

If E is the resolution of the identity for B_2, then we define

$$(4.3) \qquad S_\lambda = S_\lambda^1 + \mathfrak{R}(E_\lambda) \quad \text{and} \quad T_\lambda = T_\lambda^1 + \mathfrak{R}(I - E_\lambda),$$

where S_λ^1, T_λ^1 are obtained from Lemma 4.1, and E_λ comes from (1.2).

Theorem 4.2. *Let* P_1, P_2, *and* Q *be as in* (4.1), (4.2). *If* λ *is an* A-*positive eigenvalue or a point of continuous spectrum of* Q (*not on the left boundary*), *then there is* $\varepsilon > 0$ *such that*

$$\lambda = \min_{S \in N_\varepsilon(S_\lambda, T_\lambda)} \sup_{x \in S \cap C_+} (Bx, x)/(Ax, x),$$

where S_λ, T_λ *come from* (4.3) *and* N_ε *refers to the topology of* H_A, *i.e., of* H.

Proof. Again, we check assumptions (1) and (2).

(1) For any $x \in S_\lambda$, we write $x = x_1 + x_2$ with $x_1 \in S_\lambda^1$ and $x_2 \in \mathfrak{R}(E_\lambda)$, so

$$(Bx, x) = (Bx_1, x_1) + (Bx_2, x_2) \leq \lambda(Ax_1, x_1) + \lambda(Ax_2, x_2) = \lambda(Ax, x).$$

Furthermore, if λ is an A-positive eigenvalue with a corresponding eigenvector $x_0 \in C_+$, then we can assume without loss of generality that either $x_0 \in H_1$ or $x_0 \in H_2$. Thus either $x_0 \in S_\lambda^1$ or $x_0 \in \mathfrak{R}(E_\lambda)$, i.e., $x_0 \in S_\lambda \cap C_+$ whence

$$(4.4) \qquad \lambda = \sup_{x \in S_\lambda \cap C_+} (Bx, x)/(Ax, x).$$

If $\lambda \in \sigma_c(Q)$ but not on the left boundary, then $\lambda \in \sigma_c(B_2)$ and (1.3) holds. Arguing as for (3.1), we obtain

$$\lambda = \sup_{x \in R(E_\lambda) \cap C_+} (Bx, x)/(Ax, x),$$

which leads to (4.4) again.

(2) If λ is an A-positive eigenvalue, there is an eigenvector $x_0 \in S_\lambda \cap C_+$ as in (1). Let $T_\gamma = T_\lambda + \text{span}\{x_0\}$ for any $\gamma > 0$. Then $x_0 \in S_\lambda \cap T_\gamma$ and $T_\gamma \supset T_\lambda$. Further

$$\inf_{x \in T_\gamma \cap C_+} (Bx, x)/(Ax, x) = \lambda > \lambda - \gamma.$$

If $\lambda \in \sigma_c(Q)$ but not on the left boundary, let $T_\gamma = T_\lambda^1 + \mathfrak{R}(I - E_{\lambda-\gamma})$ for any $\gamma > 0$. Then $S_\lambda \cap T_\gamma \cap C_+ = \mathfrak{R}(E_\lambda - E_{\lambda-\gamma}) \subset H_2$ is nontrivial, so x_γ exists with $\delta = 1/2$. Further

$$\inf_{x \in T_\gamma \cap C_+} (Bx, x)/(Ax, x) \geq \lambda_0 - \gamma,$$

so, assumption (2) is also satisfied. $\square$

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