COMPOSITION OPERATORS ON POTENTIAL SPACES

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Abstrect. By a result of B. Dahlberg, the composition operators $T_Hf = H \circ f$ need not be bounded on some of the Sobolev spaces (or spaces of Bessel potentials) even for very smooth functions $H = H(t)$, $H(0) = 0$, unless of course, $H(t) = ct$. In this note a natural domain is found for $T_H$ that is, in a sense, maximal and on which the $\{T_H\}$ form an algebra of bounded operators. Here the functions $H(t)$ need not be bounded though they are required to have a sufficient number of bounded derivatives.

1. Introduction

For $H: \mathbb{R} \to \mathbb{R}$, let $T_H$ be the associated composition operator that takes $f: \mathbb{R}^n \to \mathbb{R}$ to $H(f)$. We consider the action of $T_H$ on the Bessel potential spaces $L^p_\alpha(\mathbb{R}^n)$ for $\alpha > 0$ and $1 < p < \infty$. We say that $H$ is $\alpha$-admissible if $H(0) = 0$ and

$$M \equiv \max_{k \in \mathbb{R}} \sup_{k \in \mathbb{R}} |H^{(k)}(t)| < \infty,$$

where the max is taken over $k \in \{1, \ldots, m\}$ if $\alpha = m \in \mathbb{Z}^+$, and over $k \in \{1, \ldots, m+1\}$ if $m < \alpha < m+1$, $m \in \mathbb{Z}^+$. If in addition, $H$ is bounded (an assumption not required for our results), then $T_H$ is often referred to as a smooth truncation operator; cf. [2].

Let $\dot{L}^p_\alpha$, $\alpha > 0$, $1 < p < \infty$, be the Riesz potential space, i.e. the homogeneous counterpart to the inhomogeneous space $L^p_\alpha$. Complete definitions are given below; for now we note that if $\alpha = m \in \mathbb{Z}^+$, then

$$\|f\|_{\dot{L}^p_\alpha} \approx \sum_{|\gamma| = m} \|D^\gamma f\|_{L^p},$$

while

$$\|f\|_{L^p_\alpha} \approx \sum_{|\gamma| \leq m} \|D^\gamma f\|_{L^p} \approx \|f\|_{L^p} + \|f\|_{\dot{L}^p_\alpha}.$$

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Theorem A. Suppose $\alpha > 1$, $1 < p < \infty$, and $H$ is $\alpha$-admissible. If $f \in L^p_\alpha \cap L^1_1$, then $H(f) \in L^p_\alpha \cap \dot{L}^1_1$ and
\begin{equation}
\|H(f)\|_{L^p_\alpha} \leq cM(\|f\|_{L^p_\alpha} + \|f\|_{\dot{L}^1_1}),
\end{equation}
and
\begin{equation}
\|H(f)\|_{\dot{L}^1_1} \leq M\|f\|_{L^p_\alpha}.
\end{equation}

The nonlinearity of estimate (1.1) is a reflection of the nonlinearity of the operator $T_H$; cf. (1) of [3] and (1.6) of [2].

For $0 < \alpha \leq 1$ and $1 < p < \infty$, $H(f) \in L^p_\alpha$ for all $f \in L^p_\alpha$ and any $\alpha$-admissible $H$. In fact, we see below in §3 (see (3.1), (3.2), and (3.3)) that in this case, we have
\begin{equation}
\|H(f)\|_{L^p_\alpha} \leq cM\|f\|_{L^p_\alpha}.
\end{equation}

And, in particular, it is easy to see that (1.2) holds. We also have $H(f) \in L^p_\alpha$ for all $f \in L^p_\alpha$ if $\alpha \geq n/p$, as noted in [1], or as a consequence of Theorem A, since the Sobolev (potential) imbedding theorem implies that $L^p_\alpha$ is continuously imbedded into $\dot{L}^1_1$ whenever $\alpha p \geq n$, $\alpha > 1$ and $1 < p < \infty$. However, for $\alpha \in \mathbb{Z}^+$, Dahlberg [5] has shown that if $1 < \alpha < n/p$, $1 < p < \infty$, and $H(f) \in L^p_\alpha$ for every $f \in L^p_\alpha$, $H$ admissible, then $H(f) = ct$ for some $c \in \mathbb{R}$.

In view of Dahlberg’s negative result, a natural question is to determine what extra conditions of $f \in L^p_\alpha$ guarantee that $H(f) \in L^p_\alpha$ for $\alpha$-admissible $H$. The first result of this type is obtained from the Gagliardo–Nirenberg lemma (see [7] and Lemma 2.2 below), which implies that $H(f) \in L^p_\alpha$ for every $f \in L^p_\alpha \cap L^\infty$; see [2]. This, along with an extension of the Fefferman–Stein decomposition of BMO, was used in [2] to show that $H(f) \in L^p_\alpha$ for every $f \in L^p_\alpha \cap \text{BMO}$. This result also follows from Theorem A because of the imbedding
\begin{equation}
\dot{L}^p_\alpha \cap \text{BMO} \to \dot{L}^1_1
\end{equation}
if $\alpha > 1$ and $1 < p < \infty$; see Lemma 2.2 below. It should also be noted that the methods used here to prove Theorem A are considerably simpler than those used in [2]. As for the sharpness of our main result, we also prove

Theorem B. Let $\alpha = m \in \mathbb{Z}^+$, $m \geq 1$ and suppose that $f: \mathbb{R}^n \to \mathbb{R}$ and $H(f) \in L^p_m$ for all $m$-admissible $H$. Then $f \in L^p_m \cap \dot{L}^m_1$.

In §2 we give the necessary preliminaries, definitions, and statements for the needed lemmas. The proofs of Theorems A and B are given in §3. We discuss the proofs of the lemmas in §4.

2. Preliminaries

For $\alpha > 0$, let $G_\alpha$ be the Bessel potential kernel of order $\alpha$, i.e. the positive $L^1(\mathbb{R}^n)$ function that satisfies $\hat{G}_\alpha(\xi) = (1 + |\xi|^2)^{-\alpha/2}$ for $\xi \in \mathbb{R}^n$, $\hat{\ }$ denoting the Fourier transform on $L^1$. For $1 < p < \infty$, let $L^p_\alpha = L^p_\alpha(\mathbb{R}^n)$ be the space of all $f = G_\alpha \ast g$, for some $g \in L^p(\mathbb{R}^n)$; in this case $\|f\|_{L^p_\alpha} = \|g\|_{L^p}$. The homogeneous space $\dot{L}^p_\alpha = \dot{L}^p_\alpha(\mathbb{R}^n)$ is the space of Riesz potentials $f = I_\alpha \ast g$, $g \in L^p$. Here $\hat{I}_\alpha(\xi) = |\xi|^{-\alpha}$. However, for simplicity below, we use the
equivalent Littlewood–Paley characterization of $\hat{L}_p^\alpha$ for its definition; see [8, 11].

Let $\phi: \mathbb{R}^n \to \mathbb{C}$ satisfy

\begin{align*}
(2.1) & \quad \phi \in \mathcal{S}, \\
(2.2) & \quad \text{supp} \phi \subseteq \{ \xi : 1/2 \leq |\xi| \leq 2 \}, \\
(2.3) & \quad |\hat{\phi}(\xi)| \geq c > 0 \quad \text{if } 3/5 \leq |\xi| \leq 5/3,
\end{align*}

and

\begin{equation}
(2.4) \quad \sum_{\nu \in \mathbb{Z}} \hat{\phi}(2^\nu \xi) = 1 \quad \text{for } \xi \neq 0.
\end{equation}

For $\nu \in \mathbb{Z}$, set $\phi_\nu(x) = 2^\nu \phi(2^\nu x)$. For $1 < p < \infty$ and $\alpha \geq 0$, let

$$
\|f\|_{L_p^\alpha} = \left\| \left( \sum_{\nu \in \mathbb{Z}} |2^\nu \phi_\nu \ast f|^2 \right)^{1/2} \right\|_{L_p}.
$$

When $\alpha = 0$, $L_p^\alpha = L_p$, $1 < p < \infty$ by Littlewood-Paley theory [8, 11]. With this definition, the following lemma is a simple consequence of Hölder’s inequality.

**Lemma 2.1.** Suppose $0 \leq \alpha_1 < \alpha_2 < \infty$, $0 < p_1$, $p_2 < \infty$, $0 < \theta < 1$, $\alpha = (1 - \theta)\alpha_1 + \theta \alpha_2$, and $1/p = (1 - \theta)/p_1 + \theta/p_2$. Then

$$
\|f\|_{L_p^\alpha} \leq \|f\|_{L_p^{\alpha_1}}^{1 - \theta} \|f\|_{L_p^{\alpha_2}}^\theta.
$$

In general, $L_p^\alpha$ is a space of tempered distributions modulo polynomials. However, in this paper we consider $\|f\|_{L_p^\alpha}$ only for functions $f$ belonging to $L_q$ for some $q \in (1, \infty)$. This eliminates any ambiguity regarding polynomials.

The following two facts about $L_p^\alpha$ are well known (see e.g. [8] or [11]). For $1 < p < \infty$ and $\alpha > 0$,

\begin{equation}
(2.5) \quad \|f\|_{L_p^\alpha} \approx \|f\|_{L_p} + \|f\|_{L_p^\omega},
\end{equation}

while for $1 \leq m \leq \alpha < m + 1$, $m \in \mathbb{Z}$,

\begin{equation}
(2.6) \quad \|f\|_{L_p^\alpha} \approx \sum_{|\gamma| = m} \|D^\gamma f\|_{L_p^{\alpha - m}}.
\end{equation}

Here "$\approx$" means that the ratio of the two sides is bounded above and below by finite positive constants independent of $f$. We are using the standard multi-index notation: $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\gamma_i \geq 0$, $\gamma_i \in \mathbb{Z}$, $i = 1, \ldots, n$, $|\gamma| = \gamma_1 + \cdots + \gamma_n$, and $D^\gamma = (\partial/\partial x_1)^{\gamma_1} \cdots (\partial/\partial x_n)^{\gamma_n}$. Obviously, (2.5) and (2.6) imply the classical result of Calderón [4]: if $1 < p < \infty$ and $1 \leq m \leq \alpha < m + 1$, $m \in \mathbb{Z}$, then

\begin{equation}
(2.7) \quad \|f\|_{L_p^\alpha} \approx \|f\|_{L_p} + \sum_{|\gamma| = m} \|D^\gamma f\|_{L_p^{\alpha - m}}.
\end{equation}

For $f \in L_{1 \text{loc}}(\mathbb{R}^n)$, let $\|f\|_{\text{BMO}} = \sup_Q (1/(|Q|)) \int_Q |f - f_Q| \, dx$, where $f_Q = (1/|Q|) \int_Q f$ and the supremum is over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the axes.
Lemma 2.2. Suppose $0 < \theta < 1$, $\alpha > 0$, and $1 < p < \infty$. Then
\begin{equation}
\|f\|_{L^p_{\text{BMO}}} \leq c\|f\|_{L^p_{\text{BMO}}}^{1-\theta} \cdot \|f\|_{L^p}^\theta.
\end{equation}

Clearly $\|f\|_{\text{BMO}} \leq 2\|f\|_{L^\infty}$; using this inequality in (2.8) results in the Gagliardo–Nirenberg lemma [7].

In his study of the fractional order potential spaces, Strichartz [10] introduced the operator $S_\alpha$, defined for $0 < \alpha < 1$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ by
$$S_\alpha f(x) = \left( \int_0^\infty \left[ \int_{|y|<1} |f(x + ry) - f(x)| dy \right]^{2 \alpha} r^{1-2\alpha} dr \right)^{1/2}.$$

Polking [9] introduced a class of variants of $S_\alpha$; here we consider only the special case $D_t^\alpha$, $1 \leq t < \infty$, $0 < \alpha < 1$, defined for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ by
$$D_t^\alpha f(x) = \left( \int_0^\infty \left[ \int_{|y|<1} |f(x + ry) - f(x)|^t dy \right]^{2 \alpha} r^{1-2\alpha} dr \right)^{1/2t}.$$

Obviously, $D_t^\alpha f = S_\alpha f$.

Lemma 2.3. Suppose $0 < \alpha < 1$ and $1 < p < \infty$. If $f \in L^q$ for some $q \in (1, \infty)$, then
\begin{equation}
\|f\|_{L^p_{\alpha}} \approx \|S_\alpha f\|_{L^p},
\end{equation}
and, if $1 \leq t < p$,
\begin{equation}
\|D_t^\alpha f\|_{L^p} \leq c(\alpha, p, t)\|f\|_{L^p_{\alpha}}.
\end{equation}

In (2.9) the estimate $\|f\|_{L^p_{\alpha}} \leq c(\alpha, p)\|S_\alpha f\|_{L^p}$ holds for any $f \in L^1_{\text{loc}}$; for the converse of (2.9) and (2.10), an assumption like $f \in L^q$ is needed since polynomials have norm zero in $L^q_{\alpha}$ whereas $S_\alpha$ and $D_t^\alpha$ only annihilate constants.

For $h \in \mathbb{R}^n$, set $\Delta_h f(x) = f(x + h) - f(x)$. Following Polking [9], we easily have
\begin{equation}
\Delta_h (fg)(x) = f(x) \Delta_h g(x) + g(x) \Delta_h f(x) + \Delta_h f(x) \cdot \Delta_h g(x).
\end{equation}

As a result, we can estimate the $L_{\alpha}$ norm of a product of functions in terms of their individual norms in other potential spaces. This method gives

Lemma 2.4. Suppose $0 < \theta < 1$, $1 < p < \infty$, $k \in \mathbb{Z}$, $k \geq 2$; let $p_i \geq 1$, $\gamma_i \geq 1$, $i = 1, \ldots, k$, and $1/p_j + \sum_{i \neq j} 1/\gamma_i = 1$, for $j = 1, \ldots, k$. Then
$$\left\| \prod_{i=1}^k f_i \right\|_{L_{\alpha}^\theta} \leq c \sum_{j=1}^k \|f_j\|_{L_{\alpha}^{p_j}} \cdot \prod_{i=1}^k \|f_i\|_{L_{\alpha}^{\gamma_i}}.$$

3. Proofs of the theorems

Proof of Theorem A. Since $H(0) = 0$, we have $|H(f)(x)| = |H(f(x)) - H(0)| \leq \|H'\|_{L^{\infty}} \cdot |f(x)|$. Hence
\begin{equation}
\|H(f)\|_{L^p} \leq \|H'\|_{L^{\infty}} \|f\|_{L^p}, \quad 0 < p \leq \infty.
\end{equation}
For $i = 1, \ldots, n$, let $D_i = \partial / \partial x_i$. Then $D_i H(f) = H'(f) \cdot D_i f$, and hence by (2.6),

$$\|H(f)\|_{L^p} \leq c \|H'\|_{L^\infty} \|f\|_{L^p}, \quad 1 < p < \infty. \tag{3.2}$$

If $0 < \alpha < 1$, applying the mean value theorem to $H(f(x + ry)) - H(f(x))$ gives $S_\alpha H(f) \leq \|H'\|_{L^\infty} S_\alpha(f)$. Therefore by (2.9),

$$\|H(f)\|_{L^p} \leq c \|H'\|_{L^\infty} \|f\|_{L^p} \tag{3.3}.$$

(Estimates (3.1)–(3.3) give the analogue of Theorem A for $0 < \alpha \leq 1$, $1 < p < \infty$, as noted in the introduction.)

Now, suppose that $\alpha = m \in \mathbb{Z}$, $m \geq 2$. For a multi-index $\gamma$ with $|\gamma| = m$ and $k \in \{1, \ldots, m\}$, let $A(k, \gamma)$ be the collection of all ordered $k$-tuples of multi-indices $\beta = \{\beta_1, \ldots, \beta_k\}$ with $\beta_i \in (\mathbb{Z}^+)^n$, $|\beta_i| \geq 1$, and $\sum_i \beta_i = \gamma$.

By the Leibniz rule, there exist scalars $c(\beta, k, \gamma)$ such that

$$D^\gamma H(f) = \sum_{k=1}^m H^{(k)}(f) \sum_{\beta \in A(k, \gamma)} c(\beta, k, \gamma) D^{\beta_1} f \cdots D^{\beta_k} f. \tag{3.4}$$

For $\gamma_i = m/|\beta_i|$, $|\beta_1| + \cdots + |\beta_k| = m$, Hölder’s inequality together with (2.6) gives

$$\left\| \sum_{i=1}^k D^{\beta_i} f \right\|_{L^p} \leq \prod_{i=1}^k \|D^{\beta_i} f\|_{L^{\|\beta_i\|}} \leq \prod_{i=1}^k \|f\|_{L^{2\|\beta_i\|}}. \tag{3.5}$$

By Lemma 2.1, with $t_i = (|\beta_i| - 1)/(\alpha - 1)$, $i = 1, \ldots, k$,

$$\|f\|_{L^{(m-1)/(m-1)}} \leq \|f\|_{L^{(m-1)/(m-1)}}. \tag{3.5}$$

Summing $1 - \theta_i$ and $\theta_i$, respectively, over $i = 1, \ldots, k$ gives

$$\left\| \sum_{i=1}^k D^{\beta_i} f \right\|_{L^p} \leq c \|f\|_{L_{(m-1)/(m-1)}} \left( m^{(k-1)/(m-1)} \|f\|_{L_{(m-1)/(m-1)}} \right). \tag{3.5}$$

The terms on the right side of (3.5) for $k = 1, \ldots, m$ are dominated by the term for either $k = 1$ or $k = m$ depending on whether $\|f\|_{L_{(m-1)/(m-1)}}$ exceeds $\|f\|_{L_{(m-1)/(m-1)}}$ or vice-versa. Thus by (2.6), (3.4), and (3.5),

$$\|H(f)\|_{L^p} \leq c M(\|f\|_{L^p} + \|f\|_{L^{\infty}}). \tag{3.6}$$

This with (2.5) and (3.1) establishes the desired result for $\alpha \in \mathbb{Z}^+$.

Now suppose that $1 \leq m < \alpha < m + 1$, $m \in \mathbb{Z}$. In view of (2.7), (3.1), and (3.4), fix $\gamma$ with $|\gamma| = m$, $k \in \{1, \ldots, m\}$, and $\beta \in A(k, \gamma)$. By Lemma 2.3, or more precisely, by the remark immediately following its statement,

$$\|H^{(k)}(f) D^{\beta_1} f \cdots D^{\beta_k} f\|_{L_{\alpha-m}} \leq c \|S_{\alpha-m}(H^{(k)}(f) D^{\beta_1} f \cdots D^{\beta_k} f)\|_{L^p}.$$

As noted by Polking [9], the identity

$$\Delta_h(FG)(x) = F(x + h) \Delta_h G(x) + G(x) \Delta_h F(x)$$

yields

$$S_{\theta}(FG) \leq \|F\|_{L^\infty} S_{\theta} G + |G| S_{\theta} F.$$
for $0 < \theta < 1$. Hence by (3.4)

\begin{equation}
\|D^\gamma H(f)\|_{L^p_{\alpha-m}} \leq c \sum_{k=1}^{m} \sum_{\beta \in A(k, \gamma)} (\|I\|_{L^p} + \|I\|_{L^p}),
\end{equation}

where

\[ I = D^{\beta_1} f \cdots D^{\beta_k} f \cdot S_{\alpha-m}(H^{(k)}(f)) \]

and

\[ II = \|H^{(k)}\|_{L^\infty} S_{\alpha-m}(D^{\beta_1} f \cdots D^{\beta_k} f). \]

To estimate $I$, let $r = \alpha/(\alpha - m)$ and $\gamma_i = \alpha/|\beta_i|$, $i = 1, \ldots, k$. Hölder's inequality gives

\[ \|I\|_{L^p} \leq \|D^{\beta_1} f\|_{L^{r_1}} \cdots \|D^{\beta_k} f\|_{L^{r_k}} \|S_{\alpha-m}(H^{(k)}(f))\|_{L^{r_\gamma}}, \]

since $|\beta_1| + \cdots + |\beta_k| = m$. Let $\theta_i = (|\beta_i| - 1)/(\alpha - 1)$ for $i = 1, \ldots, k$. Then by (2.6) and Lemma 2.1,

\begin{equation}
\|D^{\beta_i} f\|_{L^{r_i}} \leq c \|f\|_{L^{\gamma_i}} \leq c \|f\|_{L^\infty}^{1-\theta_i} \|f\|_{L^\infty}^{\theta_i}.
\end{equation}

Summing on $i$ yields

\[ \prod_{i=1}^{k} \|D^{\beta_i} f\|_{L^{r_i}} \leq c \|f\|_{L^{\gamma}} \leq c \|f\|_{L^\infty}^{1-\lambda} \|f\|_{L^\infty}^{\lambda}, \]

By (2.9), Lemma 2.2, and the trivial inequality $\|F\|_{BMO} \leq 2\|F\|_{L^\infty}$, we have

\[ \|S_{\alpha-m}(H^{(k)}(f))\|_{L^p} \approx \|H^{(k)}(f)\|_{L^p_{\alpha-m}} \leq c \|H^{(k)}(f)\|_{L^\infty}^{m+1-\alpha} \|H^{(k)}(f)\|_{L^p_{\alpha-m}}^{\alpha-m}. \]

Note that the equivalence above follows since constants are annihilated by both the $S_{\alpha-m}$ operator and the $L^p_{\alpha-m}$ norm, hence we can apply (2.9) to $(H^{(k)}(f) - H^{(k)}(0))$, which is in $L^p$, since $|H^{(k)}(f) - H^{(k)}(0)| \leq \|H^{(k+1)}\|_{L^\infty} \cdot |f|$. So applying (3.2) to $H^{(k)}(f)$ gives

\[ \|S_{\alpha-m}(H^{(k)}(f))\|_{L^p} \leq c M \|f\|_{L^{\alpha-m}}^{\alpha-m}, \]

and then we have

\begin{equation}
\|I\|_{L^p} \leq c M \cdot \|f\|_{L^\infty}^{(1-\lambda)} \|f\|_{L^\infty}^{\lambda},
\end{equation}

where $\lambda = (m - k)/(\alpha - 1)$.

To estimate $II$, we claim that we can apply (2.9) and conclude that

\begin{equation}
\|II\|_{L^p} \leq c M \cdot \|D^{\beta_1} f \cdots D^{\beta_k} f\|_{L^p_{\alpha-m}}.
\end{equation}

To justify this, note first that $f \in L^{\alpha}_\alpha \subset L^{\alpha}_m \subset L^{\alpha}_m$, since $\alpha > m$. Also note that $D_if \in L^{\alpha_1} \cap L^p$, $i = 1, \ldots, n$, since we are assuming $f \in L^{\alpha_1} \cap L^p$, and $\alpha > 1$. Thus it follows that $D_if \in L^{m_1}$ since $1 < m < \alpha$, or $f \in L^{m_1}$. Now applying (3.5), we have $\prod_{i=1}^{k} D^{\beta_i} f \in L^p$. 

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Now if $k = 1$ in (3.10), then $|\beta_i| = m$ and (2.6) implies $\|D^{\beta_1} f\|_{L^p_{\alpha-m}} \leq c\|f\|_{L^p_{\alpha}}$. If $2 \leq k \leq m$, we apply Lemma 2.4 with $\gamma_i = \alpha/|\beta_i|$ and $p_i = \alpha/(\alpha - m + |\beta_i|)$, $i = 1, \ldots, k$. Notice that

$$1/p_j + \sum_{i \neq j} 1/\gamma_i = \left(\alpha - m + \sum_{i=1}^k |\beta_i|\right)/\alpha = 1.$$  

Hence

$$\left|\prod_{i=1}^k D^{\beta_i} f\right|_{L^p_{\alpha-m}} \leq c_k \sum_{j=1}^k \|D^{\beta_j} f\|_{L^p_{\alpha-m}} \prod_{i \neq j} \|D^{\beta_i} f\|_{L^p_{\alpha}}.$$  

By (2.6) and Lemma 2.1,

$$\|D^{\beta_j} f\|_{L^p_{\alpha-m}} \leq c\|f\|_{L^p_{\alpha-m + |\beta_j|}} \leq c\|f\|_{L^p_{\alpha}}^{1-j} \|f\|_{L^p_{\alpha}}^{\lambda_j},$$  

where $\lambda_j = 1 - (m - |\beta_j|)/(\alpha - 1)$. We finally estimate $\|D^{\beta_j} f\|_{L^p_{\alpha}}$ by (3.8) with the same values of $\theta_i$. Altogether, this gives

(3.11) $\|\Pi f\|_{L^p} \leq cM\|f\|_{L^p}^{\alpha(\alpha - k)/(\alpha - 1)},$  

with $\alpha = (\alpha - k)/(\alpha - 1)$, which as noted above, is even correct when $k = 1$.

Putting estimates (3.9) and (3.11) into (3.7), we see that the term $cM\|f\|_{L^p}^{\alpha}$ arises in (3.9) when $k = m$, while $cM\|f\|_{L^p}^{\alpha}$ arises in (3.11) when $k = 1$. The other terms for $k \in \{1, \ldots, m\}$ in (3.9) and (3.11) are dominated by one or the other of these extreme terms, so that

$$\|D^\gamma H(f)\|_{L^p_{\alpha-m}} \leq cM\|f\|_{L^p_{\alpha}} + \|f\|_{L^p_{\alpha}}^{\alpha},$$  

for $|\gamma| = m$. With (2.5) and (3.1), the proof of Theorem A is complete.  

Proof of Theorem B. First notice that $f \in L^p_{\alpha}$ since $H(t) = t$ is $m$-admissible for any $m$. Let $\chi \in C^\infty(\mathbb{R}^n)$ satisfy $\text{supp} \chi \subseteq [-1, 1]$ and $\chi(t) = 1$ for $-2/3 \leq t \leq 2/3$. For $j = 1, \ldots, m$, let

$$H_j(t) = \sum_{l \in \mathbb{Z}} (t - 2l)^j \chi(t - 2l)$$  

and $A = \bigcup_{l \in \mathbb{Z}} (-2/3 + 2l, 2/3 + 2l)$. Also let $\tilde{H}_j(t) = H_j(t - 1)$ and $\tilde{A} = \bigcup_{l \in \mathbb{Z}} (1/3 + 2l, 5/3 + 2l)$. Clearly $A \cup \tilde{A} = \mathbb{R}$, and

(3.12) $H_j^{(j)}(t) = j!$  

for $t \in A$, and $\tilde{H}_j^{(j)}(t) = j!$  

for $t \in \tilde{A}$.

Obviously, by periodicity both $H_j$ and $\tilde{H}_j$ are $m$-admissible for all $j$ and $m$.

Now fix an $i \in \{1, \ldots, n\}$. By the chain rule

(3.13) $D_i^{(m)} H(f) = \sum_{k=1}^m H^{(k)}(f) F_k,$  

where $F_k = \sum_{m_1, \ldots, m_k} c_{m_1, \ldots, m_k} D_i^{(m_1)} f \cdots D_i^{(m_k)} f$, with the sum extending over all $m_1, \ldots, m_k$ such that $\sum_{j=1}^k m_j = m$, $m_j \geq 1$ for each $j$. (Recall $D_i = \partial/\partial x_i$). Note that $F_k$ depends on $f$ and $i$, but not on $H$. By assumption,
$D^{(m)}_i H(f) \in L^p$ for $H = H_j$ or $H = \tilde{H}_j$, $j = 1, \ldots, m$. We show by induction that this implies that $F_j \in L^p$, $j = 1, \ldots, m$. First, since we have noted that $f \in L^p$, we have $F_1 = D^{(m)}_i f \in L^p$. Now suppose $F_1, \ldots, F_{j-1} \in L^p$.

Let $H$ be $H_j$ in (3.13). By (3.12), which obviously implies that $H_j^{(k)}(t) = 0$ for $t \in A$ and $k \geq j + 1$, we have

$$D^{(m)}_i H_j(f)(x) = j! F_j(x) + \sum_{k=1}^{j-1} H_j^{(k)}(f(x)) \cdot F_k(x)$$

for $x \in f^{-1}(A)$. Since $H_j^{(k)} \in L^\infty$ and $F_1, \ldots, F_{j-1} \in L^p$, it follows that $F_j \in L^p(f^{-1}(A))$. By the same argument with $\tilde{H}_j$ in place of $H_j$, $F_j \in L^p(f^{-1}(\tilde{A}))$; hence $F_j \in L^p(\mathbb{R}^n)$. This completes the induction; after finitely many steps, we obtain that $F_m = (D_1 f)^m \in L^p$, or $D_1 f \in L^{mp}$. Since $i \in \{i, \ldots, n\}$ is arbitrary, we have $f \in L^{mp}$.

The analogue of Theorem B for $0 < \alpha \notin \mathbb{Z}^+$ remains open.

4. PROOF OF THE LEMMAS

Proof of Lemma 2.1. Applying Hölder’s inequality with exponents $1/(1 - \theta)$ and $1/\theta$ gives

$$\|f\|_{L^p_\theta} = \left\| \sum_{\nu \in \mathbb{Z}} (2^{\nu(1-\theta)}|\phi_\nu \ast f|^{1-\theta} 2^{\nu\theta} |\phi_\nu \ast f|^{\theta})^2 \right\|^{1/2}_{L^p} \leq \left\| \sum_{\nu \in \mathbb{Z}} (2^{\nu(1-\theta)}|\phi_\nu \ast f|^2)^{(1-\theta)/2} \right\|^{1/2}_{L^p} \left\| \sum_{\nu \in \mathbb{Z}} (2^{\nu\theta} |\phi_\nu \ast f|^2)^{\theta/2} \right\|^{1/2}_{L^p}.$$  

Hölder’s inequality again, this time with conjugate exponents $p_1/(1 - \theta)p$ and $p_2/\theta p$, yields the result. □

Proof of Lemma 2.2. This can be derived from the complex interpolation result $[\text{BMO}, \tilde{L}^p_{\theta}] \approx \tilde{L}^{p/\theta}_{\theta}$. Also we could give a proof based on the theory of the “$\phi$-transform” $S_{\phi}$ in [6] and the Calderón product (see [6, Theorems 2.2, 9.2, 8.2]). However, we omit the proof since all that is used in the proof of Theorem A is the well-known Gagliardo–Nirenberg inequality [7], i.e. (2.8) with $L^\infty$ in place of BMO. □

Proof of Lemma 2.3. The inhomogeneous analogues of (2.9) and (2.10), i.e.

$$\|f\|_{L^p_\theta} \approx \|f\|_{L^p} + \|S_{\alpha} f\|_{L^p} \quad \text{and} \quad \|D^\nu_{\alpha} f\|_{L^p} \leq c\|f\|_{L^p_{\theta}},$$

are proved in Strichartz [10], and Polking [9], respectively. Undoubtedly, (2.9) and (2.10) are implicit in their arguments. However, we want to note another proof based on the Littlewood–Paley definition of $\tilde{L}^p_{\theta}$ given in §2. The analogue of (2.10) in the inhomogeneous case for $t = 1$ is done in Triebel [11, Theorem 2.5.11]. With a few modifications, the same argument gives (2.10). The assumption $f \in L^q$ is used to prove the pointwise a.e. equality of $f$ and $\sum_n \phi_\nu \ast f$ under assumptions (2.1)–(2.4). We omit the details.

We do, however, give a proof of the lower bound of $S_{\alpha} f$ in (2.9). This proof appears to be more direct than the corresponding proofs contained in the
literature. For \( r > 0 \), let
\[
A_{r, \alpha} f(x) = r^{-\alpha} \int_{|y| < 1} |f(x + ry) - f(x)| \, dy
\]
\[
= r^{-\alpha - n} \int_{|y| < r} |f(x + y) - f(x)| \, dy.
\]
It is easy to see that if \( 2^\mu \leq r \leq 2^{\mu+1} \), then
\[
2^{-\alpha - n} A_{2^\mu, \alpha} f(x) \leq A_{r, \alpha} f(x) \leq 2^{\alpha + n} A_{2^{\mu+1}, \alpha} f(x).
\]
Thus a simple discretization of the integral from 0 to \( \infty \) shows that
\[
(4.1) \quad S_\alpha f(x) \approx \left( \sum_{\mu \in \mathbb{Z}} |A_{2^\mu, \alpha} f(x)|^2 \right)^{1/2},
\]
with constants depending only on \( n \) and \( \alpha \).

By (2.2), \( \int \phi = 0 \), so for \( \nu \in \mathbb{Z} \),
\[
\phi_\nu * f(x) = \int [f(x - y) - f(x)] \phi_\nu(y) \, dy
\]
\[
= \int [f(x - 2^{-\nu} y) - f(x)] \phi(y) \, dy.
\]
For \( \mu \in \mathbb{Z} \), let \( R_\mu = \{ x \in \mathbb{R}^n: 2^{\mu-1} \leq |x| \leq 2^\mu \} \) and \( B_\mu = \{ x \in \mathbb{R}^n: |x| \leq 2^\mu \} \).
Selecting \( M > n + \alpha \), we have
\[
2^{\nu \alpha} |\phi_\nu * f(x)| = 2^{\nu \alpha} \sum_{\mu \in \mathbb{Z}} \int_{R_\mu} |\phi(y)| \cdot |f(x - 2^{-\nu} y) - f(x)| \, dy
\]
\[
\leq 2^{\nu \alpha} \sum_{\mu \in \mathbb{Z}} \sup_{R_\mu} |\phi| \cdot \int_{B_\mu} |f(x - 2^{-\nu} y) - f(x)| \, dy
\]
\[
\leq c_M 2^{\nu \alpha} \sum_{\mu \in \mathbb{Z}} (1 + 2^\mu)^{-M} 2^{\nu n} \int_{B_\mu-\nu} |f(x + y) - f(x)| \, dy
\]
\[
= c_M \sum_{\mu \in \mathbb{Z}} 2^{\mu(n+\alpha)} (1 + 2^\mu)^{-M} \cdot A_{2^{\mu-\nu}, \alpha} f(x).
\]
This last sum is a discrete convolution and \( \{ 2^{\mu(n+\alpha)} (1 + 2^\mu)^{-M} \} \in l^1(\mathbb{Z}) \), so Young's inequality gives
\[
\left( \sum_{\nu \in \mathbb{Z}} |2^{\nu \alpha} \phi_\nu * f(x)|^2 \right)^{1/2} \leq c_{\mu, \alpha} \left( \sum_{\mu \in \mathbb{Z}} |A_{2^\mu, \alpha} f(x)|^2 \right)^{1/2}.
\]
Taking \( L^p \)-norms and using (4.1) gives \( \| f \|_{L^p_G} \leq c \| S_\alpha f \|_{L^p} \). \( \Box \)

**Proof of Lemma 2.4.** We first prove the case \( k = 2 \). Using (2.11) and applying the Cauchy–Schwarz inequality twice to the final term, gives
\[
S_\theta(f_1,f_2)(x) \leq \| f_1(x) \| S_\theta f_2(x) + \| f_2(x) \| S_\theta f_1(x) + D_2^{\theta/2} f_1(x) D_2^{\theta/2} f_2(x).
\]
(This is a special case of Polking's formula in [9].) Hence by (2.9) and the remark following Lemma 2.3,
\[
\| f_1 f_2 \|_{L^p} \leq c \| f_1 S_\theta f_2 \|_{L^p} + c \| f_2 S_\theta f_1 \|_{L^p} + c \| D_2^{\theta/2} f_1 D_2^{\theta/2} f_2 \|_{L^p}.
\]
Since $1/p_1 + 1/\gamma_2 = 1$, Hölder’s inequality and (2.9) imply
\[ \|f_2 S_\theta f_1\|_{L^p} \leq \|S_\theta f_1\|_{L^{p_1}} \|f_2\|_{L^{\gamma_2 p}} \leq c \|f_1\|_{L^{p_1}} \|f_2\|_{L^{\gamma_2 p}}. \]
(Note that (2.9) and (2.10) apply to $f_1$ and $f_2$ since $f_j \in L^{\gamma_j p}$, $j = 1, 2$, else there is nothing to prove.) Similarly,
\[ \|f_1 S_\theta f_2\|_{L^p} \leq c \|f_2\|_{L^{p_2}} \|f_1\|_{L^{\gamma_1}}. \]
Define conjugate indices $s$ and $s'$ so that $1/s = (1/\gamma_1 + 1/p_1)/2$ and $1/s' = (1/\gamma_2 + 1/p_2)/2$. By Hölder’s inequality and (2.10),
\[ \|D_2^{\theta/2} f_1 D_2^{\theta/2} f_2\|_{L^p} \|D_2^{\theta/2} f_1\|_{L^{p_1}} \|D_2^{\theta/2} f_2\|_{L^{s' p}} \leq c \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}. \]
By Lemma 2.1, we have
\[ \|f_1\|_{L^{p_1}} \leq \|f_1\|_{L^{s'}}^{1/2} \|f_1\|_{L^{p_2}}^{1/2}, \]
and
\[ \|f_2\|_{L^{s' p}} \leq \|f_2\|_{L^{p_1}}^{1/2} \|f_2\|_{L^{p_2}}^{1/2}. \]
Thus since $|abcd|^{1/2} \leq |ab| + |cd|$, we have
\[ \|f_1 f_2\|_{L^p} \leq c(\|f_1\|_{L^{p_1}} \|f_2\|_{L^{\gamma_2 p}} + \|f_2\|_{L^{p_2}} \|f_1\|_{L^{\gamma_1}}), \]
as desired in the case $k = 2$. The general case follows by induction on $k$. We only sketch the details. Apply the case $k = 2$ with $p_1$, $\gamma_2$, $p_2$, $\gamma_1$, $f_1$, $f_2$ replaced by $\gamma_k' = \gamma_k/(\gamma_k - 1)$, $\gamma_k$, $p_k$, $p_k' = p_k/(p_k - 1)$, $\prod_{i=1}^{k-1} f_i$, and $f_k$, respectively. This yields an estimate for $\|\prod_{i=1}^{k} f_i\|_{L^p}$ involving two products of the form $\prod_{i=1}^{k-1}$. Using the $k - 1$ case for one, and Hölder’s inequality for the other, yields the result. □

References


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