PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 114, Number 1, January 1992

BIFURCATION OF LIMIT CYCLES: GEOMETRIC THEORY

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(Communicated by Kenneth R. Meyer)

ABSTRACT. Multiple limit cycles play a basic role in the theory of bifurcations. In this paper we distinguish between singular and nonsingular, multiple limit cycles of a system defined by a one-parameter family of planar vector fields. It is shown that the only possible bifurcation at a nonsingular, multiple limit cycle is a saddle-node bifurcation and that locally the resulting stable and unstable limit cycles expand and contract monotonically as the parameter varies in a certain sense. Furthermore, this same type of geometrical behavior occurs in any one-parameter family of limit cycles experiencing a saddle-node type bifurcation except possibly at a finite number of points on the multiple limit cycle.

1. INTRODUCTION

This paper contains some basic new information concerning bifurcations at multiple limit cycles of a planar system

$$\dot{x} = f(x, \lambda) \qquad (1_\lambda)$$

depending on a parameter $\lambda \in \mathbf{R}$. It is assumed that f is an analytic function; although, in discussing bifurcations at a limit cycle of multiplicty m, it suffices to assume that f is of class C^m.

We distinguish between two types of multiple limit cycles, singular and nonsingular multiple limit cycles, and prove that the bifurcation theory for nonsingular, multiple limit cycles is exactly the same as the bifurcation theory for multiple limit cycles belonging to a one-parameter family of vector fields; cf. [5, 9]. In other words, the only possible type of bifurcation that can occur at a nonsingular, multiple limit cycle Γ_0 of (1_λ) is a saddle-node bifurcation in which Γ_0 bifurcates into two hyperbolic limit cycles, one stable and the other unstable, which expand and contract monotonically as the parameter λ varies in a certain sense (described in Table 1 in §2). We also prove that this type of geometrical behavior occurs in any one-parameter family of limit cycles that experiences a saddle-node type of bifurcation except possibly at a finite number of points on the multiple limit cycle.

The proofs, as well as the definition of a singular, multiple limit cycle, are based on certain properties of the Poincaré map; they utilize the Weierstrass

Received by the editors March 25, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58F14, 58F21.
This work was supported by a Northern Arizona University Organized Research Grant.

preparation theorem and some of the same techniques, based on Puiseux series, that were employed by the author in [8].

2. Bifurcation at nonsingular multiple limit cycles

Assume that the system (1_λ) has a limit cycle

$$\Gamma_0 \colon x = \gamma_0(t)$$

at some parameter value, say $\lambda = 0$, and let T_0 denote the minimum period of the periodic function $\gamma_0(t)$. For convenience in notation, we define the function $f_0(t) = f(\gamma_0(t), 0)$, and let $\nabla \cdot f_0(t)$ denote the function $\nabla \cdot f(\gamma_0(t), 0)$. We also define ω_0 to be ± 1 according to whether Γ_0 is positively or negatively oriented respectively. For planar systems, we utilize the following classical definition of a multiple limit cycle; cf. [1, p. 118].

Definition 1. The limit cycle Γ_0 is a *multiple limit cycle of* (1_0) if

$$\int_0^{T_0} \nabla \cdot f_0(t)\, dt = 0.$$

In order to understand the meaning of this definition and, indeed, to understand the theory of bifurcation of limit cycles, one must be familiar with the Poincaré map. This idea originated with Poincaré [10] at the turn of the century. In order to define the Poincaré map, let l_τ be the straight line normal to the limit cycle Γ_0 at the point $\gamma_0(\tau) \in \Gamma_0$ and let s denote the signed distance along l_τ with $s > 0$ on the exterior of Γ_0 and $s < 0$ on the interior of Γ_0. The existence and analyticity of the Poincaré map $h(s, \lambda, \tau)$ for $|s| < \delta$, $|\lambda| < \delta$, $\tau \in \mathbf{R}$, and some $\delta > 0$ is established in [1, 7]. Cf. [8, Lemma 1.1 and Remark 1.1]. In terms of the Poincaré map, $h(s, \lambda, \tau)$, we define the *displacement function*

$$d(s, \lambda, \tau) = h(s, \lambda, \tau) - s.$$

See Figure 1. If the point $\gamma_0(\tau)$ on Γ_0 plays no role in a particular use of the displacement function $d(s, \lambda, \tau)$, then we simply denote the displacement function by $d(s, \lambda)$.

The following result is classical (cf. [1, p. 383]).

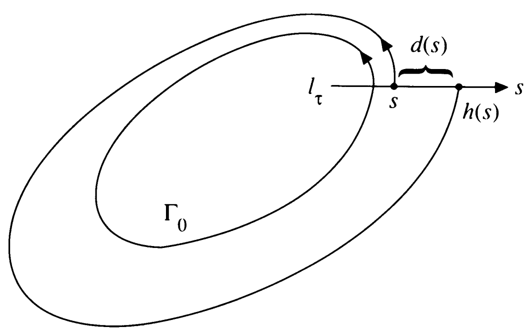


FIGURE 1. (The Poincaré map and displacement function for (1_λ) in a neighborhood of the limit cycle Γ_0 of (1_0).)

Lemma 1. *For $\delta > 0$, $|s| < \delta$, and $|\lambda| < \delta$, let $d(s, \lambda)$ denote the displacement function for the system (1_λ). Then the derivative $d_s(0, 0)$ is independent of the point $\gamma_0(\tau)$ on Γ_0 and*

$$d_s(0, 0) = e^{\int_0^{T_0} \nabla \cdot f_0(t)\, dt} - 1. \tag{2}$$

We see that Γ_0 is a 8multiple limit cycle of (1_0) if and only if for all $\tau \in \mathbf{R}$, that $d_s(0, 0, \tau) = 0$; and this is equivalent to saying that Γ_0 has a characteristic multiplier equal to one (cf. [7]).

Definition 2. If

$$d(0, 0) = d_s(0, 0) = d_s^{(2)}(0, 0) = \cdots = d_s^{(m-1)}(0, 0) = 0$$

and

$$d_s^{(m)}(0, 0) \neq 0$$

then, for $m > 1$, Γ_0 is called a multiple limit cycle of *multiplicity* m. If $m = 1$, i.e., if $d(0, 0) = 0$ and $d_s(0, 0) \neq 0$, then Γ_0 is called a *simple limit cycle* or hyperbolic limit cycle.

Remark 1. If follows from Theorem 42 [1, p. 277] that the multiplicity m of a multiple limit cycle Γ_0 is independent of the point $\gamma_0(\tau)$ on Γ_0; in fact, according to Theorem 42 in [1], the multiplicity m of Γ_0 is equal to the maximum number of limit cycles that can bifurcate from Γ_0 under a perturbation of (1_0). And for analytic systems,

$$d(0, 0) = d_s(0, 0) = d_s^{(2)}(0, 0) = \cdots = 0$$

if and only if Γ_0 is a cycle that belongs to a continuous band of cycles (cf. [1] or [7]).

As was pointed out by Chicone and Jacobs [3], one of the most important results for planar systems (1_λ), depending on a parameter $\lambda \in \mathbf{R}$, is the following result, which is established in [1, p. 384]. (The *wedge product* of two vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ is defined by $x \wedge y = x_1 y_2 - y_1 x_2$).

Lemma 2. *For $\delta > 0$, $|s| < \delta$, $|\lambda| < \delta$, and $\tau \in \mathbf{R}$, let $d(s, \lambda, \tau)$ denote the displacement function for the system (1_λ) along the normal line l_τ and let ω_0 denote the orientation of Γ_0. Then*

$$d_\lambda(0, 0, 0) = -\frac{\omega_0 e^{\int_0^{T_0} \nabla \cdot f_0(t)\, dt}}{|f_0(0)|} \int_0^{T_0} e^{-\int_0^t \nabla \cdot f_0(u)\, du} f \wedge f_\lambda(\gamma_0(t), 0)\, dt. \tag{3}$$

First we extend the result in Lemma 2 and determine the dependence of $d(0, 0, \tau)$ on τ, i.e., on the point $\gamma_0(\tau) \in \Gamma_0 \cap l_\tau$.

Lemma 3. *Under the hypotheses of Lemma 2,*

$$\begin{aligned} d_\lambda(0, 0, \tau) &= -\frac{\omega_0 e^{\int_\tau^{\tau+T_0} \nabla \cdot f_0(t)\, dt}}{|f_0(\tau)|} \int_\tau^{\tau+T_0} e^{-\int_\tau^t \nabla \cdot f_0(u)\, du} f \wedge f_\lambda(\gamma_0(t), 0)\, dt. \\ &= -\frac{\omega_0 e^{\int_0^{\tau+T_0} \nabla \cdot f_0(t)\, dt}}{|f_0(\tau)|} \int_\tau^{\tau+T_0} e^{-\int_0^t \nabla \cdot f_0(u)\, du} f \wedge f_\lambda(\gamma_0(t), 0)\, dt. \end{aligned} \tag{4}$$

This lemma is proved by replacing the functions $\gamma_0(t)$ and $f_0(t)$ in equation (3) by the functions $\gamma_0(t + \tau)$ and $f_0(t + \tau)$ respectively and then showing that the result reduces to equation (4).

Remark 2. The integral containing the wedge product in equation (4) is related to the Melnikov function, which plays such an important role in the theory of perturbed dynamical systems (cf. [6 or 7]). We define the function

$$M(\tau) = \int_{\tau}^{\tau+T_0} e^{-\int_\tau^t \nabla\cdot f_0(u)\,du} f \wedge f_\lambda(\gamma_0(t), 0)\,dt$$

for later use.

Our first new result, which forms the basis for much of this paper, is the following:

Theorem 1. *Under the hypotheses of Lemma* 2, *if* Γ_0 *is a multiple limit cycle of* (1_0), *then*

$$d_\lambda(0, 0, \tau) = \frac{|f_0(0)|}{|f_0(\tau)|} e^{\int_0^\tau \nabla\cdot f_0(t)\,dt} d_\lambda(0, 0, 0).$$

Proof. By definition, Γ_0 is a multiple limit cycle if and only if

$$\int_0^{T_0} \nabla \cdot f_0(t)\,dt = 0.$$

Thus, since $\nabla \cdot f_0(t)$ is a T_0-periodic function of mean value zero, it follows that

$$\int_0^{t+T_0} \nabla \cdot f_0(u)\,du = \int_0^t \nabla \cdot f_0(u)\,du.$$

And then, since $f \wedge f_\lambda(\gamma_0(t), 0)$ is a T_0-periodic function of t, it follows that the integrand in expression (4) in Lemma 3 is a T_0-periodic function of t. Therefore, it follows from Lemmas 2 and 3 that

$$\begin{aligned} d_\lambda(0, 0, \tau) &= -\frac{\omega_0 e^{\int_0^{\tau+T_0} \nabla\cdot f_0(t)\,dt}}{|f_0(\tau)|} \int_\tau^{\tau+T_0} e^{-\int_0^t \nabla\cdot f_0(u)\,du} f \wedge f_\lambda(\gamma_0(t), 0)\,dt \\ &= -\frac{\omega_0 e^{\int_0^{\tau} \nabla\cdot f_0(t)\,dt}}{|f_0(\tau)|} \int_0^{T_0} e^{-\int_0^t \nabla\cdot f_0(u)\,du} f \wedge f_\lambda(\gamma_0(t), 0)\,dt \\ &= \frac{|f_0(0)|}{|f_0(\tau)|} e^{\int_0^\tau \nabla\cdot f_0(t)\,dt} d_\lambda(0, 0, 0). \end{aligned}$$

Corollary 1. *Under the hypotheses of Theorem* 1, $d_\lambda(0, 0, 0)$ *is positive, zero, or negative if and only if for all* $\tau \in \mathbf{R}$, $d_\lambda(0, 0, \tau)$ *is positive, zero, or negative respectively.*

Definition 3. The limit cycle Γ_0 is a *singular, multiple limit cycle of* (1_0) if

$$d_s(0, 0) = 0 \quad \text{and} \quad d_\lambda(0, 0) = 0.$$

If $d_s(0, 0) = 0$ and $d_\lambda(0, 0) \neq 0$, then Γ_0 is a *nonsingular, multiple limit cycle of* (1_0).

Remark 3. Note that Γ_0 is a singular, multiple limit cycle of (1_0) if and only if

$$\int_0^{T_0} \nabla \cdot f_0(t)\,dt = 0$$

and

$$\int_0^{T_0} e^{-\int_0^t \nabla\cdot f_0(u)\,du} f \wedge f_\lambda(\gamma_0(t), 0)\,dt = 0.$$

Definition 4. $\mu(\tau) = \operatorname{sgn} M(\tau) = \operatorname{sgn} \int_{\tau}^{\tau+T_0} e^{-\int_0^t \nabla \cdot f_0(u)\, du} f \wedge f_\lambda(\gamma_0(t), 0)\, dt$.

Corollary 2. *Under the hypotheses of Theorem* 1, *if* Γ_0 *is a nonsingular, multiple limit cycle of* (1_0), *then for all* $\tau \in \mathbf{R}$, $\mu(\tau) = \mu(0)$.

Since, by definition, any one-parameter family of rotated vector fields $f(x, \lambda)$ satisfies

$$f \wedge f_\lambda(x, \lambda) > 0$$

(cf. [5] or [9]). We immediately obtain the following:

Theorem 2. *If* $f(x, \lambda)$ *defines a one-parameter family of rotated vector fields in* $\mathbf{R}^2$ *with parameter* $\lambda \in \mathbf{R}$, *then any multiple limit cycle of* (1_λ) *is a nonsingular, multiple limit cycle of* (1_λ).

In order to present our main result for nonsingular, multiple limit cycles, we define σ_0 to be ± 1 according to whether the limit cycle Γ_0 is unstable or stable on its exterior respectively; and we define $\mu_0 = \mu(0)$ where $\mu(\tau)$ is defined in Definition 4. Then if Γ_0 is a nonsingular, multiple limit cycle of (1_0), it follows from Corollary 2 that for all $\tau \in \mathbf{R}$, $\mu(\tau) = \mu_0$.

Our main result for nonsingular, multiple limit cycles is a simple application of the implicit function theorem (since $d_\lambda(0, 0) \neq 0$) (cf. [8]). The following result generalizes the results for one-parameter families of rotated vector fields contained in Theorem 8 in [5], the results contained in Theorem F in [9], and the results contained in Theorems 71, 72 in [1]. It shows that the behavior at a nonsingular, multiple limit cycle of (1_λ) is exactly the same as the behavior at a multiple limit cycle in a one-parameter family of rotated vector fields; i.e., the only bifurcation that can occur at a nonsingular, multiple limit cycle Γ_0 is a saddle-node bifurcation where Γ_0 splits into two simple limit cycles of the opposite stability.

Theorem 3. *If* Γ_0 *is a nonsingular, multiple limit cycle of* (1_0), *then* Γ_0 *belongs to a unique, one-parameter family of limit cycles of* (1_λ) *and*

(1) *if the multiplicity of* Γ_0 *is odd, then the family either expands or contracts monotonically as* λ *increases through zero as determined by Table* 1 *while*

(2) *if the multiplicity of* Γ_0 *is even, then* Γ_0 *bifurcates into a simple stable limit cycle and a simple unstable limit cycle as* λ *varies in a certain sense, determined by Table* 1, *and* Γ_0 *disappears as* λ *varies in the opposite sense.*

Remark 4. The stable and unstable limit cycles generated in a saddle-node bifurcation at a nonsingular, multiple limit cycle Γ_0 of even multiplicity, m, also

ω_0	$+$	$+$	$-$	$-$
σ_0	$+$	$-$	$+$	$-$
$\mu_0\Delta\lambda$	$+$	$-$	$-$	$+$

Table 1. The change in λ, $\Delta\lambda$, that causes the expansion of a nonsingular, multiple limit cycle Γ_0 of odd multiplicity or the bifurcation of a nonsingular, multiple limit cycle Γ_0 of even multiplicity.

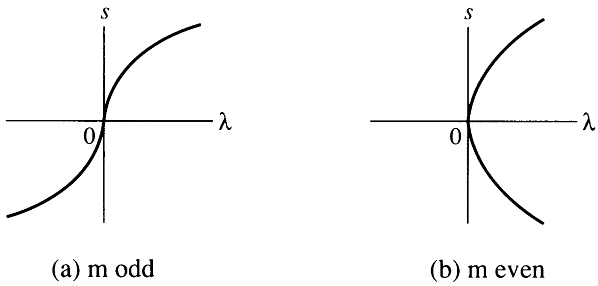


FIGURE 2. The curve $d(s, \lambda) = 0$ describing a one-parameter family of limit cycles of (1_λ).

expand or contract monotonically for sufficiently small λ as described by Table 1. For example, if Γ_0 is a positively oriented, nonsingular, multiple limit cycle that is unstable on its exterior (i.e., $\omega_0 = +1$ and $\sigma_0 = +1$), then for $\mu_0 = +1$, it follows from Table 1 that Γ_0 bifurcates into a simple stable limit cycle Γ_λ^- and a simple unstable limit cycle Γ_λ^+ as the parameter λ increases. Furthermore, for sufficiently small $\lambda > 0$, $\Gamma_\lambda^- \subset \operatorname{Int}\Gamma_0$, $\Gamma_\lambda^+ \subset \operatorname{Ext}\Gamma_0$ and in this case Γ_λ^+ expands monotonically and Γ_λ^- contracts monotonically away from Γ_0 as λ increases from zero. The bifurcation diagram for this case is shown in Figure 2(b). When Γ_0 is a nonsingular, multiple limit cycle, the curve $d(s, \lambda) = 0$ is described by a function $\lambda = \lambda(s) = s^m(b_0 + b_1 s + b_2 s^2 + \cdots)$ for $|s| < \delta$ and some $\delta > 0$ according to the implicit function theorem. Although the coefficients $b_i = b_i(\tau)$ depend on the point $\gamma_0(\tau) \in \Gamma_0 \cap l_\tau$, $b_0(\tau) \neq 0$ for $\tau \in \mathbf{R}$ since $d_\lambda(0, 0, \tau) \neq 0$ for $\tau \in \mathbf{R}$; in fact, $\operatorname{sgn}[b_0(\tau)] = \omega_0\sigma_0\mu_0$ for all $\tau \in \mathbf{R}$. In the case when $\omega_0\sigma_0\mu_0 = +1$, the nature of the curve $d(s, \lambda) = 0$ for m odd or even is shown in Figure 2(a) or 2(b) respectively. In case $\omega_0\sigma_0\mu_0 = -1$, the curve $d(s, \lambda) = 0$ can be obtained by rotating the curves shown in Figure 2 about the s-axis.

If Γ_0 is a simple limit cycle, then at any point $\gamma_0(\tau)$ on Γ_0, $d_s(0, 0, \tau) \neq 0$ and it follows from the implicit function theorem that for each $\tau \in \mathbf{R}$ the curve $d(s, \lambda, \tau) = 0$ is described by a function $s = s(\lambda, \tau)$ and the rate of growth of the limit cycle Γ_0 with resepct to λ at the point $\gamma_0(\tau) \in \Gamma_0$ is determined by

$$\frac{\partial s}{\partial \lambda}(0, \tau) = -\frac{d_\lambda(0, 0, \tau)}{d_s(0, 0, \tau)}$$

(cf. [5, equation (3.17)]). It follows from Lemmas 1 and 3 that

$$\operatorname{sgn}\left[\frac{\partial s}{\partial \lambda}(0, \tau)\right] = \omega_0\sigma_0\mu(\tau),$$

since a simple limit cycle is stable (or unstable) if and only if

$$\int_0^{T_0} \nabla \cdot f_0(t)\, dt < 0 \quad (\text{or } > 0),$$

i.e., $\operatorname{sgn}[d_s(0, 0)] = \sigma_0$ (cf. [1, p. 118] or [7, Chapter 3, §4]). These results can be summarized as follows:

Theorem 4. *If Γ_0 is a simple limit cycle of (1_0), then Γ_0 belongs to a unique, one-parameter family of limit cycles Γ_λ of (1_λ) and at any point $\gamma_0(\tau)$ on Γ_0,* *increasing the parameter* λ *causes the limit cycle* Γ_λ *to expand or contract along the normal line* l_τ *if and only if* $\omega_0\sigma_0\mu(\tau) = \pm 1$ *respectively.*

An example of a one-parameter family of limit cycles Γ_λ that expands along various portions of Γ_0 and contracts along other portions of Γ_0 is furnished by the last example in §2 of [8]. Of course, if $\mu(\tau)$ is of one sign, as in the case when $f(x, \lambda)$ defines a one-parameter family of rotated vector fields, then according to Theorem 4, the simple limit cycle Γ_0 expands or contracts monotonically as λ increases through zero as determined by Table 1. This generalizes the results for families of rotated vector fields contained in Theorem 7 in [5] and Theorem D in [9].

3. Bifurcation at singular multiple limit cycles

In §2 we saw that if Γ_0 is a simple limit cycle or a nonsingular, multiple limit cycle of (1_0), then there is only one branch of $d(s, \lambda) = 0$ passing through the origin and the only bifurcation that can occur at Γ_0 is a saddle-node bifurcation. On the other hand, if Γ_0 is a singular, multiple limit cycle of multiplicity m, then it follows from the Weierstrass preparation theorem that there may be as many as m branches of $d(s, \lambda) = 0$ passing through the origin (cf. [8]).

If Γ_0 is a singular, multiple limit cycle of (1_0) that belongs to a one-parameter family of limit cycles Γ_λ, then as in [8], Γ_λ is defined by a branch $s(\lambda, \tau)$ of $d(s, \lambda, \tau) = 0$, which can be expanded in a Puiseux series

$$s(\lambda, \tau) = (\sigma\lambda)^{k/m} \sum_{i=0}^{\infty} a_i(\tau)(\sigma\lambda)^{i/m} \tag{5}$$

for $0 \le \sigma\lambda < \sigma\delta$ where σ is ± 1, $a_0(\tau) \ne 0$ except possibly at finitely many $\tau \in [0, T_0)$, δ is some positive constant and k and m are unique relatively prime positive integers (cf. [1] or [4]). In this case Γ_0 is said to be a multiple limit cycle belonging to a one-parameter family of limit cycles Γ_λ of *reduced multiplicity* m. For simplicity in notation, we consider the point $\tau \in [0, T_0)$ as fixed in what follows and delete the dependence of the displacement function on τ even though the coefficeints $a_i(\tau)$ in the Puiseux series (5) can depend on τ; cf. Remark 5 below.

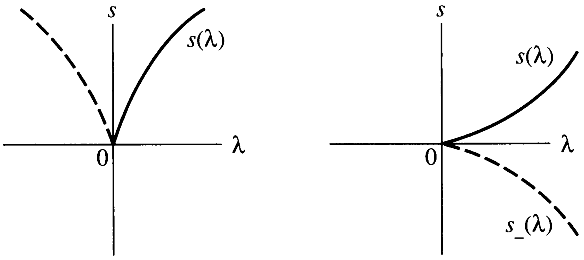
In order to illustrate the type of geometrical information that can be obtained from the Puiseux series (5), consider the case when $\sigma = +1$ and $a_0 > 0$. For m odd, let $\beta = \lambda^{1/m}$. Then there is a unique analytic continuation of the function

$$s(\beta^m) = \beta^k \sum_{i=0}^{\infty} a_i \beta^i,$$

defined by (5) for $0 \le \beta < \delta$, to an interval $|\beta| < \delta$. And for m even, k is odd, and the inverse of the function $s(\lambda)$ exists and satisfies

$$\lambda(s) = s^{m/k} \sum_{i=0}^{\infty} b_i s^{i/k} \tag{6}$$

for $0 \le s < \delta$, where $b_0 > 0$ and δ is some positive constant. Then since k is odd, just as above, this function can be unqiuely continued to a function defined by (6) on an interval $|s| < \delta$. Now let $\lambda = \beta^m$ and $\beta = \pm\lambda^{1/m}$ with



(a) m odd (b) m even

FIGURE 3. Typical branches of $d(s, \lambda) = 0$ along a fixed normal line l_τ to the limit cycle Γ_0.

$\operatorname{sgn}(\beta) = \operatorname{sgn}(s)$. Then from (6),

$$\beta = s^{1/k}\left[\sum_{i=0}^{\infty} b_i s^{i/k}\right]^{1/m}$$

for $|s| < \delta$. Since k is odd, this function is one-to-one for $|s| < \delta$ and its inverse is the unique analytic continuation of the function $s(\beta^m)$, defined by the Puiseux series (5) for $0 \le \beta < \delta$, to an interval $|\beta| < \delta$. This is equivalent to continuing the upper branch of $s(\lambda)$ given by (5) to the lower branch given by

$$s_-(\lambda) = -\lambda^{k/m}\sum_{i=0}^{\infty}(-1)^i a_i \lambda^{i/m} \tag{7}$$

for $0 \le \lambda < \delta$; and this continuation is unique. Note that for $\sigma = +1$, the functions $s(\lambda)$, $\lambda(s)$ and $s_-(\lambda)$, defined by (5), (6), and (7) respectively, satisfy $s(\lambda(x)) = x$, $s_-(\lambda(-x)) = -x$ and $\lambda(s(x)) = \lambda(s_-(x)) = x$ for all x in $[0, \delta)$. A typical bifurcation diagram for the branch $s(\lambda)$ of $d(s, \lambda) = 0$ and its continuation $s_-(\lambda)$ when m is even is shown in Figure 3(b). A typical branch $s(\lambda)$ of $d(s, \lambda) = 0$ and its continuation when m is odd is shown in Figure 3(a).

Similar results follow when σ or a_0 is negative and this leads to the following result:

Theorem 5. *Suppose that Γ_0 is a singular, multiple limit cycle of (1_0) that belongs to a one-parameter family of limit cycles Γ_λ of (1_λ), corresponding to a branch $s(\lambda, \tau)$ of $d(s, \lambda, \tau) = 0$, of reduced multiplicity m. Then for all but possibly a finite number of $\tau \in [0, T_0)$, there is a $\delta > 0$ such that $s(\lambda, \tau)$ can be expanded in a Puiseux series, (5) with $a_0(\tau) \ne 0$, which converges for $0 \le \sigma\lambda < \sigma\delta$ where $\sigma = \pm 1$; furthermore,*

(1) *if m is even, then Γ_0 bifurcates into a simple stable limit cycle and a simple unstable limit cycle belonging to the family Γ_λ as $\sigma\lambda$ increases and Γ_0 disappears as $\sigma\lambda$ decreases;*

(2) *if m is odd and k is odd, then the limit cycles of the family Γ_λ expand or contract along the normal line l_τ to Γ_0 as $\sigma\lambda$ increases according to whether $a_0(\tau)$ is positive or negative respectively; and*

(3) *if m is odd and k is even, then the limit cycles of the family Γ_λ expand or contract along the normal line l_τ to Γ_0 as λ increases in $(0, \delta)$ or*

as λ decreases in $(-\delta, 0)$ according to whether $a_0(\tau)$ is positive or negative respectively.

Corollary 3. *Under the hypotheses of Theorem 5, if the reduced multiplicity m is even, the leading coefficient $a_0(\tau)$ in the Puiseux series (5) does not change sign.*

This corollary follows immediately from Theorem 5 since for m even, the simple unstable and stable limit cycles $\Gamma_\lambda^\pm$ in part (1) of Theorem 5 correspond to the branches $s_\pm(\lambda, \tau)$ or to the branches $s_\mp(\lambda, \tau)$ of $d(s, \lambda, \tau) = 0$ where $s_\pm(\lambda, \tau)$ are defined by (5) and (7) respectively. But then if $a_0(\tau)$ changes sign, this would imply that for all sufficiently small $\sigma\lambda > 0$ the limit cycles Γ_λ^+ and Γ_λ^- of (1_λ) intersect, a contradiction. Thus, $a_0(\tau)$ does not change sign.

Remark 5. The relatively prime integers m and k in the Puiseux series (5) are unique and independent of the point $\gamma_0(\tau) \in \Gamma_0$ through which the normal line l_τ passes. In fact, all possible values of k/m are given by the absolute values of the slopes of the sides of the Newton polygon for the displacement function $d(s, \lambda, \tau)$ as described in [1] or [4]. However, the coefficients $a_0(\tau)$ can depend on τ and for m odd, $a_0(\tau)$ can change sign as τ varies in $[0, T_0)$. Thus, Theorem 5 tells us whether at particular point $\gamma_0(\tau) \in \Gamma_0$, the family of limit cycles Γ_λ moves inward or outward along the normal line l_τ as the parameter λ increases through zero. It does not imply that the family Γ_λ expands or contracts monotonically in a neighborhood of Γ_0 unless $a_0(\tau)$ is of a constant sign. Since the coefficient $a_0(\tau)$ is either very difficult or impossible to determine in terms of the vector field $f(x, \lambda)$ and the periodic orbit $\gamma_0(t)$, Theorem 5 does not appear to be of much practical value; however, it is the basis for the theorem establishing that any one-parameter family of limit cycles can be continued through a bifurcation in a unique way, proved in [8], as well as for the next geometrical theorem.

Theorem 6. *Under the hypotheses of Theorem 5, if m is even, it follows that Γ_0 bifurcates into a simple stable limit cycle Γ_λ^- and a simple unstable limit cycle Γ_λ^+ as the parameter λ varies in one sense and that Γ_0 disappears as λ varies in the opposite sense. Furthermore, except at a finite number of points on Γ_0, either Γ_λ^+ expands monotically away from Γ_0 and Γ_λ^- contracts monotonically away from Γ_0 as the parameter λ varies monotonically in a neighborhood of zero or vica versa.*

Proof. If the reduced multiplicity m in equation (5) is even, then at any point $\gamma_0(\tau) \in \Gamma_0 \cap l_\tau$ where $a_0(\tau) \neq 0$ the graph of the continuation of the inverse function $\lambda(s)$, given by equation (6), for $|s| < \delta$ has the form of one of the functions (of s) shown in Figure 4 (or one of these graphs rotated about the s-axis).

According to Theorem 5, for m even, Γ_0 bifurcates into a simple stable limit cycle Γ_λ^- and a simple unstable limit cycle Γ_λ^+ as $\sigma\lambda$ increases. And, according to Corollary 3, $a_0(\tau)$ does not change sign. Thus, since $a_0(\tau)$ is analytic, $a_0(\tau)$ is either positive or negative except possibly at a finite number of points in $[0, T_0]$. Assume that $a_0(\tau) > 0$ except possibly at a finite number of points in $[0, T_0]$ and that the simple unstable and stable limit cycles $\Gamma_\lambda^\pm$ in part (1) of Theorem 5 correspond to the branches $s_\pm(\lambda, \tau)$ of $d(s, \lambda, \tau) = 0$

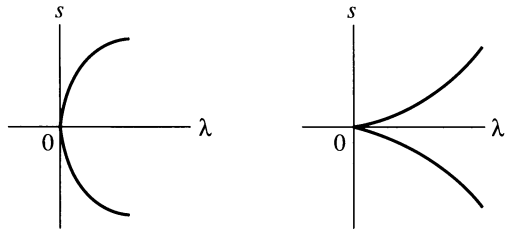


FIGURE 4. The graph of the function $\lambda(s)$ for $|s| < \delta$.

defined by (5) and (7) respectively. It then follows from the nature of the graphs of the functions $s_{\pm}(\lambda)$, shown in Figure 4 (for $\sigma = +1$), that at any point $\gamma_0(\tau) \in \Gamma_0 \cap l_\tau$, except possibly at the points $\gamma_0(\tau^*)$ where $a_0(\tau^*) = 0$, Γ_λ^+ expands monotonically away from Γ_0 and Γ_λ^- contracts monotonically away from Γ_0 as $\sigma\lambda$ increases monotonically from zero. The other cases, where $a_0(\tau) < 0$ except possibly at a finite number of points in $[0, T_0]$ or where $\Gamma_\lambda^\pm$ correspond to the branches $s_\mp(\lambda, \tau)$ respectively, can be treated similarly, and therefore, we have the conclusion of Theorem 6.

Remark 6. Theorem 6 implies that at every point $\gamma_0(\tau)$ on Γ_0, except possibly at a finite number of points $\gamma_0(\tau^*)$ where $a_0(\tau^*) = 0$, the simple unstable and stable limit cycles $\Gamma_\lambda^\pm$ expand and contract monotonically away from Γ_0 as $\sigma\lambda$ increases from zero. On the other hand, both Γ_λ^+ and Γ_λ^- may expand (or contract) monotonically away from Γ_0 in a neighborhood of a point $\gamma_0(\tau^*)$ where $a_0(\tau^*) = 0$ as shown in Figure 5.

For example, suppose that for $0 \le \lambda < \delta$

$$s_+(\lambda, \tau) = \lambda^{3/2}[a_0(\tau) + a_1(\tau)\lambda^{1/2} + \cdots].$$

Then, according to equation (7),

$$s_-(\lambda, \tau) = -\lambda^{3/2}[a_0(\tau) - a_1(\tau)\lambda^{1/2} + \cdots]$$

for $0 \le \lambda < \delta$ and if $a_0(\tau^*) = 0$, it follows that

$$s_\pm(\lambda, \tau^*) = \lambda^2[a_1(\tau^*) \pm a_2(\tau^*)\lambda^{1/2} + \cdots]$$

for $0 \le \lambda < \delta$. The nature of the functions $s_\pm(\lambda, \tau^*)$ is then shown in Figure

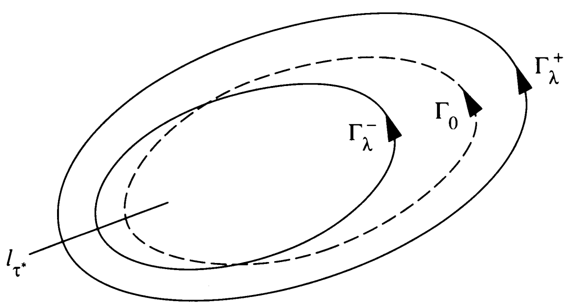


FIGURE 5. An example of a limit cycle Γ_λ^-, which neither exapnds nor contracts monotonically away from Γ_0 with a variation of the parameter λ.

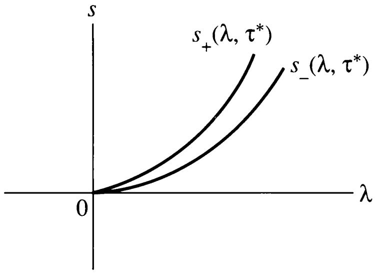


FIGURE 6. The graphs of the branches $s_\pm(\lambda, \tau^*)$ of $d(s, \lambda, \tau^*) = 0$ at a point $\gamma_0(\tau^*) \in \Gamma_0 \cap l_{\tau^*}$ where $a_0(\tau^*) = 0$.

6 (for $a_1(\tau^*) > 0$ and $a_2(\tau^*) > 0$) and the behavior of the limit cycles $\Gamma_\lambda^\pm$ near the point $\gamma_0(\tau^*) \in \Gamma_0 \cap l_{\tau^*}$ is shown in Figure 5 in this case.

The type of behavior described in Remark 6 and Figure 5 apparently occurs near the points $\gamma_0(\pm\tau^*)$ where the y-axis intersects the semistable limit cycle Γ_0, which occurs at the bifurcation value $\lambda_0 = 6.557853\ldots$, of the system

$$\dot{x} = -y + \lambda x - (1+\lambda)x^3/3 + x^5/5 \qquad \dot{y} = x.$$

(A study of this sytem was suggested by the referee.) This system is symmetric with respect to the origin and a numerical study of this system shows that the unstable limit cycle Γ_λ^+ expands monotonically away from Γ_0 at all points on Γ_0 and that the stable limit cycle Γ_λ^- contracts monotonically away from Γ_0 except in a neighborhood of the points $\gamma_0(\pm\tau^*)$ where, as λ increases from λ_0, Γ_λ^- expands monotonically away from Γ_0 (as in Figure 5). It is conjectured that $a_0(\pm\tau^*) = 0$ where $\gamma_0(\pm\tau^*)$ are the points where Γ_0 intersects the y-axis in this example.

4. CONCLUSIONS

To summarize, any bifurcation at a periodic orbit of a planar analytic system (1_λ) depending on a parameter $\lambda \in \mathbf{R}$ occurs at a multiple limit cycle of (1_λ) or at a cycle belonging to a continuous band of cycles of (1_λ). The only bifurcation that occurs at a nonsingular, multiple limit cycle is the saddle-node bifurcation with the stable and unstable bifurcating limit cycles expanding and contracting monotonically. On the other hand, as many as m one-parameter families of limit cycles can bifurcate from a singular, multiple limit cycle of multiplicity m; however, any bifurcating one-parameter family whose reduced multiplicity is even corresponds to a saddle-node type of bifurcation that exhibits the same geometrical behavior as in the nonsingular case except possibly at a finite number of points on the multiple limit cycle. It is conjectured that this same type of geometrical behavior characterizes the saddle-node type of bifurcation in any number of dimensions.

It was shown by the author in [8] that at most a finite number of one-parameter families of limit cycles can bifurcate from any cycle belonging to a continuous band of cycles and it can be shown, using the Weierstrass preparation theorem and Puiseux series, exactly as in this paper, that even in this case any bifurcating one-parameter family of even reduced multplicity corresponds to a saddle-node type of bifurcation whose limit cycles exhibit the same kind of monotone expansion and contraction as described in Theorem 6. Several interesting examples of one-parameter families of limit cycles bifurcating from a center have recently been studied by Chicone and Jacobs [3] and Blows and Perko [2].

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