THE C¹ CLOSING LEMA FOR ENDOMORPHISMS WITH FINITELY MANY SINGULARITIES

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Abstract. The C¹ closing lemma for endomorphisms with finitely many singularities is obtained by combining the C¹ closing lemma for nonsingular endomorphisms together with a technique of L. S. Young.

1. Introduction

Let \( M \) be a compact Riemannian manifold without boundary, and let \( F \) be the set of \( C¹ \) maps of \( M \) with finitely many singularities, endowed with the \( C¹ \) topology. In this paper we prove the following:

Theorem A. Let \( f \) be a \( C¹ \) map of \( M \) with finitely many singularities, and \( \omega \) be a nonwandering point of \( f \). Then for any \( C¹ \) neighborhood \( \mathcal{U} \) of \( f \) in \( F \), there is a \( g \in \mathcal{U} \) such that \( \omega \) is a periodic point of \( g \).

Recall that a point \( x \) is a singularity of \( f \) if \( T_x f \) is not injective. A point \( x \) is nonwandering of \( f \) if for any neighborhood \( U \) of \( x \) in \( M \), \( (f^n U) \cap U \) is nonempty for some \( n \geq 1 \), and periodic of \( f \) if \( f^n x = x \) for some \( n \geq 1 \).

The proof of Theorem A is based on the \( C¹ \) closing lemma of nonsingular endomorphisms [11] on the one hand, and a technique of L. S. Young [12] on the other.

2. Preliminaries

In this section we collect from [11] some definitions and theorems needed in this paper.

By a tree \( \mathcal{T} = (Q, f) \) we mean an infinite sequence of mutually disjoint nonempty finite sets \( Q_0, Q_1, \ldots, Q_n, \ldots \), where \( Q_0 \) consists of a single point \( q_0 \), together with a map \( f : Q - \{q_0\} \to Q \), where \( Q = \bigcup_{n=0}^{\infty} Q_n \), such that \( f \) maps \( Q_n \) into \( Q_{n-1} \) for each \( n = 1, 2, \ldots \). An infinite sequence \( q_0, q_1, \ldots, q_n, \ldots \) is called an infinite branch of \( \mathcal{T} \) if \( f(q_n) = q_{n-1} \) for each \( n = 1, 2, \ldots \). A finite sequence \( q_0, q_1, \ldots, q_k \) is called a finite branch of \( \mathcal{T} \) if \( f(q_n) = q_{n-1} \) for each \( n = 1, 2, \ldots, k \), and if \( f^{-1}\{q_k\} \) is empty. A tree

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\( \mathcal{F} = (Q, f) \) is called complete if \( f \) is onto. Clearly, \( \mathcal{F} \) is complete iff \( \mathcal{F} \) has only infinite branches. Occasionally, we may talk about finite trees and their branches. The definitions are obvious. Here we only mention that a finite tree is called complete if all its branches have the same number of terms.

By a tree of isomorphisms we mean a collection of linear isomorphisms parametrized by a tree \( \mathcal{F} \). More precisely, this means that we associate to each \( q \in Q \) an \( m \)-dimensional inner product space \( V_q \), and to each \( q \neq q_0 \) a linear isomorphism \( T_q : V_q \to V_{q_0} \).

The main result needed in this paper is the following \( \varepsilon \)-kernel avoiding transition theorem of [11]:

**Theorem 2.1.** Given a complete tree of isomorphisms \( (\mathcal{F}, T_q) \) and \( \varepsilon > 0 \). There is a number \( p > 2 \) and an integer \( \mu \geq 1 \) such that: For any finite ordered set \( P = \{p_0, p_1, \ldots, p_t\} \) in \( V_{q_0} \), there is a point \( y \in P \cap (B(p_t, p|p_0 - p_t|)) \) such that for any branch \( \Sigma = \{q_0, q_1, \ldots, q_n, \ldots\} \) of \( \mathcal{F} \), there is a point \( w \in P \cap (B(p_t, p|p_0 - p_t|)) \), where \( w \) is before \( y \) in the order of \( P \), together with \( \mu + 1 \) points \( c_0, c_1, \ldots, c_\mu \) in \( B(p_t, p|p_0 - p_t|) \), not necessarily distinct, satisfying the following two conditions (a) and (b).

(a) \( c_0 = w \), \( c_\mu = y \), and 
(b) \[ |T_{q_n}^{-1}(c_n) - T_{q_n}^{-1}(c_{n+1})| \leq \varepsilon d(T_{q_n}^{-1}(c_n+1), T_{q_n}^{-1}(A)) \] for \( n = 0, 1, \ldots, \mu - 1 \), where \( T_{q_0} \) stands for the identity, \( A = (P(w, y)) \cup (\partial B(p_t, p|p_0 - p_t|)) \), \( P(w, y) = \{p \in P|p \text{ is after } w \text{ and before } y\} \), and \( d \) is the distance on \( V_{q_0} \). Q.E.D.

We also need the following two basic perturbation lemmas that deal with the \( \varepsilon \)-kernel lifts and the local linearizations for \( C^1 \) maps. They are essentially the same as Lemmas 4.1 and 4.2 of [11] with some minor changes. Let \( C^1(M) \) be the set of \( C^1 \) maps of \( M \) into itself. For simplicity we assume that \( M \) is Riemann embedded into some \( R^d \). Then \( C^1(M) \) has a \( C^1 \) metric \( d_1 \) inherited from \( C^1(M, R^d) \) compatible with its \( C^1 \) topology. Fix a \( \zeta > 0 \) such that \( \exp_p \) embeds \( \{u \in T_p M | |u| \leq \zeta \} \) into \( M \) for each \( p \in M \).

**Lemma 2.2.** For any \( \eta > 0 \), there is an \( \varepsilon > 0 \) such that for any \( f \in C^1(M) \), any \( p \in M \), and any two points \( v_1, v_2 \in T_p M \) with \( B(v_2, |v_1 - v_2|/\varepsilon) \subset \{u \in T_p M | |u| \leq \zeta \} \), there is a diffeomorphism \( h = h_p, \varepsilon, v_1, v_2 : M \to M \), called an \( \varepsilon \)-kernel lift, such that:

1. \( h(\exp_p(v_2)) = \exp_p(v_1) \);
2. \( \text{supp}(h) \subset \exp_p(B(v_2, |v_1 - v_2|/\varepsilon)) \), here the support means the closure of the set where \( h \) differs from the identity;
3. \( d_1(hf, f) < \eta \). Q.E.D.

Now let \( f \in C^1(M) \) be given. Before stating Lemma 2.3, recall that the negative orbit of \( p \in M \) under \( f \) is defined as \( \text{Orb}^{-}(p) = \text{Orb}^{-}(p) = \bigcup_{n=1}^{\infty} f^{-n}\{p\} \), where \( f^{-n}\{p\} \) denotes the preimage of \( (f^n)^{-1}\{p\} \). Given an integer \( \mu \geq 1 \), if \( \bigcup_{n=1}^{\mu} f^{-n}\{p\} \) contains no singularities of \( f \), then \( f \) is a local diffeomorphism near each \( q \in \bigcup_{n=1}^{\mu} f^{-n}\{p\} \). Thus \( f^{-n}\{p\} \) is finite for each \( n = 1, 2, \ldots, \mu \), as \( M \) is compact. Assume further that all terms in \( \bigcup_{n=0}^{\mu} f^{-n}\{p\} \) are distinct. In this case we may find a neighborhood \( W \) of \( p \) in \( M \), called a \( \mu \)-dynamical neighborhood of \( p \), such that each connected component \( U \) of
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\[ \bigcup_{n=0}^{\mu} f^{-n}(W) \]

is a neighborhood of a unique point \( q \in \bigcup_{n=0}^{\mu} f^{-n}\{p\} \), denoted as \( U = W(q) = W_f(q) \) and called the \( W\)-component at \( q \), and that \( f^n \) maps \( W(q) \) onto \( W \) whenever \( f^n(q) = p \), \( n = 1, 2, \ldots, \mu \). Notice that \( \bigcup_{n=0}^{\mu} f^{-n}\{p\} \) forms in this case a finite tree, which may not be complete.

In [11, Lemma 4.2], where \( f \) was a nonsingular endomorphism, we did a local linearization along a set of the form \( \bigcup_{n=0}^{\mu} f^{-n}\{p\} \), which was a finite complete tree. In the present paper, as noticed above, \( \bigcup_{n=0}^{\mu} f^{-n}\{p\} \) is still a finite tree, but may not be complete. Although a local linearization can also be made along a finite noncomplete tree in general, for our purpose, we do a local linearization along a subset of it, which does form a finite complete tree. This is the following

**Lemma 2.3.** Let \( f \in C^1(M) \), \( p \in M \), and an integer \( \mu \geq 1 \) be given. Assume that all terms of \( \bigcup_{n=0}^{\mu+1} f^{-n}\{p\} \) are distinct and are not singularities of \( f \). Let \( Q_0, Q_1, \ldots, Q_{\mu+1} \) be nonempty sets in \( M \) such that \( Q_0 = \{p\} \), and that \( f(Q_n) = Q_{n-1} \) for each \( n = 1, 2, \ldots, \mu + 1 \). Then for any \( \eta > 0 \), there is a \( \lambda > 0 \), and a map \( f_1 \in C^1(M) \), called a local linearization of \( f \), with the following properties (1)–(5).

Write \( W = \{u \in T_p M | u \leq \lambda\} \), \( V = \{u \in T_p M | u \leq \lambda/4\} \), \( W = \exp_p(W'), V = \exp_p(V') \).

1. \( W \) is \((\mu + 1)\)-dynamical for both \( f \) and \( f_1 \), and the \( W\)-component for \( f \) and for \( f_1 \) are the same, i.e. \( W_f(q) = W_{f_1}(q) \), for each \( q \in \bigcup_{n=0}^{\mu+1} Q_n \);
2. \( f_1 = \exp_{f(q)}(T_q f) \exp_q^{-1} \) on \( V_f(q) \) if \( q \in \bigcup_{n=0}^{\mu+1} Q_n \);
3. \( f_1^{\mu+1} = f^{\mu+1} \) on \( W(q) \) if \( q \in Q_{\mu+1} \). In particular, if \( q \in Q_{\mu+1} \) then \( f_1 = \exp_{f(q)}(T_q f) f^{\mu+1} \exp_q^{-1} f^{\mu+1} \) on \( V(q) \). Note that \( V_f(q) = V_{f_1}(q) \) here and we have written both of them as \( V(q) \);
4. \( f_1 = f \) on \( M - \bigcup\{W(q) | q \in \bigcup_{n=0}^{\mu+1} Q_n\} \);
5. \( d_1(f_1, f) < \eta \). Q.E.D.

Roughly, \( f_1 \) near \( q \in \bigcup_{n=1}^{\mu+1} Q_n \) is just \( T_q f \) module those \( \exp \), and \( f_1 \) near \( q \in Q_{\mu+1} \) cancels out these linearizations. The following corollary is clear.

**Corollary 2.4.** If \( x \in W \), \( f^k x \in W(q) \) for some \( q \in f^{-\mu-1}\{p\} \), and if the orbit \( f x, f^2 x, \ldots, f^{k-1} x \) never meets \( W(q) \) for all \( q \in \bigcup_{n=1}^{\mu+1}(f^{-n}\{p\} - Q_n) \), then \( f^k x = f^k x_1 \). Q.E.D.

Roughly, this is because whenever the orbit \( f x, f^2 x, \ldots, f^{k-1} x \) meets the area of changes, they meet at a whole \( \mu + 1 \) successive iterates from some \( W(q), q \in Q_{\mu+1} \), to \( W(p) \). Thus those changes cancel out.

3. **Proof of Theorem A**

First we prove an easy lemma that is used in Case 1 of the proof of Theorem A. Let \( \Omega(f) \) and \( P(f) \) denote the sets of nonwandering points and periodic points of \( f \) respectively.

**Lemma 3.1.** Let \( f \in C^1(M) \), \( \sigma \in \Omega(f) - P(f) \). If \( \text{Orb}^{-}(\sigma) \) contains no singularities of \( f \), then \( \text{Orb}^{-}(\sigma) \cap \Omega(f), f \) forms a complete tree.

**Proof.** First we remark that if \( p \in \Omega(f) \), and if \( f^{-1}\{p\} \) contains no singularities of \( f \), then \( f^{-1}\{p\} \cap \Omega(f) \) is nonempty. Actually, since \( p \) is nonwandering, there is a sequence \( \{x_k\} \) in \( M \) and a sequence of positive integers...
\( \{n_k\} \) such that \( \{x_k\} \) and \( \{f^{n_k}(x_k)\} \) both converge to \( p \). The set \( \{f^{n_k-1}(x_k)\} \) has some limit point \( q \) since \( M \) is compact. Clearly, \( q \in f^{-1}\{p\} \). Since \( q \) is not a singularity of \( f \), \( f \) is a local diffeomorphism near \( q \). It follows that \( q \in \Omega(f) \). This proves the remark.

Now let \( Q_n = f^{-n}\{\sigma\} \cap \Omega(f) \), \( n = 0, 1, 2, \ldots \). It follows from the above remark that \( Q_n \) is nonempty for each \( n \geq 0 \). It also follows that \( f \) maps \( Q_n \) onto \( Q_{n-1} \) for each \( n \geq 1 \). Now \( \sigma \) is not periodic of \( f \), hence all terms in \( \text{Orb}^{-}(\sigma) \) are distinct. Therefore \( (\text{Orb}^{-}(\sigma) \cap \Omega(f), f) \) forms a complete tree. Q.E.D.

Now we prove Theorem A.

**Proof of Theorem A.** Let \( U \) be a small neighborhood of \( \omega \) in \( M \). It suffices to find a \( g \in \mathcal{U} \), such that \( g \) has a periodic point in \( U \), since another perturbation allows us to push this periodic point onto \( \omega \) (see [2, 5]). We assume that \( \omega \) is not periodic already of \( f \), and divide the proof into two cases.

Case 1. There is a \( \sigma \in \text{Orb}^{-}(\omega) \cap \Omega(f) \) such that \( \text{Orb}^{-}(\sigma) \) does not contain singularities of \( f \).

Assume that \( f^s(\sigma) = \omega \). Let \( N \) be a neighborhood of \( \sigma \) such that \( f^s(N) \subset U \). Since \( \omega \) is nonperiodic of \( f \), so is \( \sigma \). Then by Lemma 3.1, \( (\text{Orb}^{-}(\sigma) \cap \Omega(f), f) \) forms a complete tree. The proof in Case 1 is very much similar to the proof of Theorem B of [11]. The difference is that \( \text{Orb}^{-}(\sigma) \) this time is just a tree that may not be complete. For explicitly we write down the proof in all details as follows.

Take any \( \eta > 0 \) such that the \( \eta \)-ball of \( f \) in \( F_{\text{End}}(M) \) is contained in \( \mathcal{U} \). By Lemma 2.2, there is an \( \epsilon > 0 \) such that

\[
d_1(hf, f) < \eta/2
\]

for any \( f \in F_{\text{End}}(M) \), where \( h \) is any \( \epsilon \)-kernel lift.

Denote by \( \mathcal{T} \) the complete tree \( (\text{Orb}^{-}(\sigma) \cap \Omega(f), f) \), and denote by \( (\mathcal{T}, T_q) \) the tree of isomorphisms, where \( q \in \text{Orb}^{-}(\sigma) \cap \Omega(f) - \{\sigma\} \), and \( T_q = T_qf^n \) if \( f^nq = \sigma \).

Let \( \rho > 2, \mu \geq 1 \) be the two numbers guaranteed by Theorem 2.1 respecting \( \{\mathcal{T}, T_q\} \) and \( \epsilon > 0 \). For the \( f, \sigma, \mu \), there is by Lemma 2.3 a \( \lambda > 0 \) and a local linearization \( f_1 \) with the following properties (1)–(5). Write \( W'' = \{u \in T_\sigma M : |u| \leq \lambda\} \), \( V' = \{u \in T_\sigma M : |u| \leq \lambda/4\} \), \( W = \exp_\sigma(W'') \), \( V = \exp_\sigma(V') \).

(1) \( W \) is \( (\mu + 1) \)-dynamical for both \( f \) and \( f_1 \), and \( W_f(q) = W_{f_1}(q) \) for each \( q \in \bigcup_{n=1}^{\mu+1} f^{-n}\{\sigma\} \cap \Omega(f) \);

(2) \( f_1 = \exp_{f_1}(T_qf) \exp_q^{-1} \) on \( V_f(q) \) if \( q \in \bigcup_{n=1}^\mu f^{-n}\{\sigma\} \cap \Omega(f) \);

(3) \( f^{n+1}_1 = f^{n+1} \) on \( W(q) \) if \( q \in f^{-\mu-1}\{\sigma\} \cap \Omega(f) \). In particular, if \( q \in f^{-\mu-1}\{\sigma\} \cap \Omega(f) \), then \( f_1 = \exp_{f_1}(T_qf) f^{\mu+1} \exp_q^{-1} \) on \( V(q) \);

(4) \( f_1 = f \) on \( M - \bigcup\{W(q) : q \in \bigcup_{n=1}^{\mu+1} f^{-n}\{\sigma\} \cap \Omega(f) \} \);

(5) \( d_1(f_1, f) < \eta/2 \).

Clearly, \( f_1 \in F_{\text{End}}(M) \).

By shrinking \( W \) if necessary, we assume that \( W \subset N \), and that \( f^k(W(q)) \cap (W(q)) = \emptyset \) for any \( k \geq 1 \) and any \( q \in \bigcup_{n=1}^{\mu+1} f^{-n}\{\sigma\} \). Put a metric \( d'' \) on \( W \) by defining

\[
d''(p, q) = |u - v|,
\]
where $p, q \in W$, $u = \exp_{-1}(p)$, $v = \exp_{-1}(q)$. Since $\sigma$ is nonwandering of $f$, there are two points $p$ and $f^\psi(p)$, where $\psi \geq 1$ is an integer, such that the ball $B(f^\psi(p), pd'(p, f^\psi(p)); d')$ is contained in $V$. Let $P = \{p, fp, \ldots, f^\psi p\} \cap V$. Say, $P = \{p_0, p_1, \ldots, p_t\}$. Note that $p_0 = p, p_t = f^\psi p$. Hence $B(p_t, pd'(p_0, p_t); d') \subset V$. Let $P' = \exp_{-1}(P), p'_t = \exp_{-1}(p_t)$. Then $P' = \{p'_0, p'_1, \ldots, p'_t\}$.

By Theorem 2.1, there is a point $y' \in P' \cap B(p'_t, \rho|p'_0 - p'_t|$) such that for any branch $\Sigma = \{q_0, q_1, \ldots, q_n, \ldots\}$ of $\Orb_?(\sigma) \cap \Omega(f)$, there is a point $w'(\Sigma) \in P' \cap B(p'_t, \rho|p'_0 - p'_t|$), where $w'(\Sigma)$ is before $y'$ in $P'$, together with $\mu + 1$ points $c'_0(\Sigma), c'_1(\Sigma), \ldots, c'_\mu(\Sigma)$ in $B(p'_t, \rho|p'_0 - p'_t|$), not necessarily distinct, satisfying the following two conditions (a) and (b).

(a) $c'_0(\Sigma) = w'(\Sigma)$, $c'_\mu(\Sigma) = y'$; and
(b) $|\left( (T_{q_n}f^n)^{-1}(c'_n(\Sigma)) - (T_{q_n}f^n)^{-1}(c'_{n+1}(\Sigma)) \right) | \\
\leq \varepsilon d((T_{q_n}f^n)^{-1}(c'_0(\Sigma)), (T_{q_n}f^n)^{-1}(A))$

for $n = 0, 1, 2, \ldots, \mu - 1$, where $A = (P'(w'(\Sigma), y')) \cup \partial B(p'_t, \rho|p'_0 - p'_t|)$, and $P'(w'(\Sigma), y') = \{p \in P' \mid p$ is before $y'$ and after $w'(\Sigma)\}$.

Let $w(\Sigma) = \exp_{a}(w'(\Sigma)), y = \exp_{a}(y')$. Then $w(\Sigma)$ and $y$ are both in $P$. Hence there is an integer $\phi(\Sigma) \geq 1$ such that $f^\phi(\Sigma)(w(\Sigma)) = y$. Note that the orbit $f(w(\Sigma)), f^2(w(\Sigma)), \ldots, f^\phi(\Sigma)(w(\Sigma))$ never meets $W(q)$ for $q \in \bigcup_{n=1}^{\mu + 1} f^{-n}\{\sigma\} - \Omega(f)$ since, for those $q$, $W(q)$ is wandering (i.e. $f^k(W(q)) \cap W(q) = \varnothing$ for any $k \geq 1$). Thus it must meet a $W(q)$ for some $q \in f^{-\mu - 1}\{\sigma\} \cap \Omega(f)$, since $\Orb_?(\sigma) \cap \Omega(f)$ is a complete tree. Hence $\phi(\Sigma) > \mu + 1$. Let $z(\Sigma) = f^{\phi(\Sigma) - \mu - 1}(w(\Sigma))$. It is ready to see that $z(\Sigma)$ actually does not depend on $\Sigma$, since $w(\Sigma)$ and $y$ are both in $P$, and $f^{\phi(\Sigma)}(w(\Sigma)) = y$, and since $\mu$ and $y$ do not depend on $\Sigma$. Thus we simply write

$z = f^{\phi(\Sigma) - \mu - 1}(w(\Sigma))$

for any branch $\Sigma$. Clearly, $f^{\mu + 1}(z) = y$.

Since $y \in V$, there is a unique point $\sigma_{\mu+1}$, here $\sigma_{\mu+1}$ is in $f^{-\mu - 1}\{\sigma\} \cap \Omega(f)$ as just noticed above, such that

$z \in V(\sigma_{\mu+1})$.

Let $\Gamma$ be any branch of $\Orb_?(\sigma) \cap \Omega(f)$ that contains $\sigma_{\mu+1}$. Say $\Gamma = \{\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots\}$. Let $w'$, together with $c'_0, c'_1, \ldots, c'_\mu$, be guaranteed by Theorem 2.1 respecting $\Gamma$, and let $\phi > \mu + 1$ be the integer such that $f^\phi(w) = y$, where $w = \exp_{a}(w')$. Note that $w \in N$.

For each $\sigma_n$, $n = 0, 1, \ldots, \mu + 1$, let $h_{\sigma_n}$ be the $\varepsilon$-kernel lift obtained by treating in Lemma 2.2 $p = \sigma_n, v_1 = (T_{\sigma_n}f^n)^{-1}(c'_n)$, and $v_2 = (T_{\sigma_n}f^n)^{-1}(c'_{n+1})$. Define a map $g$ by

$g = \begin{cases} h_{\sigma_n} \circ f^\phi & \text{on } W(\sigma_{n+1}), \ n = 0, 1, \ldots, \mu - 1, \\ f^\phi & \text{on the rest of } M. \end{cases}$

Then $q \in F \End^{1}(M)$ and $d_1(g, f_1) < \eta/2$. Hence

$d_1(g, f) < \eta$. 

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We now verify that $w$ is periodic of $g$. It suffices to verify that $g^{\phi - \mu - 1}(w) = z$ and $g^{\mu + 1}(z) = w$. By the condition (b) above, the $g$ orbit from $w$ to $z$ never touches the supports of these lifts. Then $g^{\phi - \mu - 1}(w) = f_1^{\phi - \mu - 1}(w)$. But $f_1^{\phi - \mu - 1}(w) = f^{\phi - \mu - 1}(w)$ by Corollary 2.4. Therefore,

$$g^{\phi - \mu - 1}(w) = f^{\phi - \mu - 1}(w) = z.$$  

It remains to verify that $g^{\mu + 1}(z) = w$. By the condition (3) above,

$$g(z) = f_1(z) = \exp_{\sigma}(T_\sigma f^\mu)^{-1} \exp_{\sigma}^{-1} f^{\mu + 1}(z)$$

because $z \in V(\sigma_{\mu + 1})$. Hence

$$g(z) = \exp_{\sigma}(T_\sigma f^\mu)^{-1}(y)$$

because $f^{\mu + 1}(z) = y$. Thus these lifts $h_{\sigma_{\mu - 1}}, h_{\sigma_{\mu - 2}}, \ldots, h_{\sigma_0}$ give rise to

$$g^\mu(g(z)) = w$$

by condition (2) above. This verifies that $w$ is periodic of $g$.

Now since $\sigma$ is not periodic of $f$, we may take $W$ so small in advance that $f^n(W)$ does not intersect $\bigcup\{W(p) | p \in \bigcup_{n=1}^{\infty} f^{-n}(\sigma)\}$ for $n = 1, 2, \ldots, s$, here $s$ is the same integer that appeared in the beginning of Case 1 such that $f^s(\sigma) = \omega$. Thus $g^s(w) = f^s(w)$ is a periodic point of $g$ in $U$. This proves Theorem A for Case 1.

Case 2. No such $\sigma$ exists.

First we note that there is an $n_0 \geq 1$ such that $f^{-n}(\omega)$ does not contain singularities of $f$ for $n \geq n_0$. This is because otherwise there would be a singularity $p$ of $f$ and two integers $1 \leq n_1 < n_2$ such that $p \in (f^{-n_1}(\omega)) \cap (f^{-n_2}(\omega))$ because $f$ has only finitely many singularities. Then

$$f^{n_2-n_1}(\omega) = f^{n_2-n_1}(f^{n_1}(p)) = f^{n_2}(p) = \omega,$$

contradicting that $\omega$ is not periodic of $f$.

Thus by the assumption of Case 2, there is a $k \geq 1$, such that $(f^{-k}(\omega)) \cap (\Omega(f)) = \emptyset$ but $(f^{-k+1}(\omega)) \cap (\Omega(f)) \neq \emptyset$. Then we can take a $q \in (f^{-k+1}(\omega)) \cap \Omega(f)$ such that $(f^{-1}(q)) \cap \Omega(f) = \emptyset$. We now adopt a technique of [12] to handle this case. Also see [9] for this technique.

Because $q$ is nonwandering of $f$, there is a sequence of points $x_1, x_2, \ldots, x_i, \ldots$ in $M$ that converges to $q$, and a sequence of positive integers $1 < j_1 < j_2 < \cdots < j_i < \cdots$ such that the sequence $f^{j_i}(x_i)$ also converges to $q$. Let $p$ be a limit point of the sequence

$$f^{j_1}(x_1), f^{j_1}(x_2), \ldots, f^{j_i}(x_i), \ldots.$$

Then $p \in f^{-1}(q)$. Thus $p \notin \Omega(f)$.

Fix a ball $B$ around $p$ such that $(f^n B) \cap B = \emptyset$ for all $n \geq 1$. Since $q \in \Omega(f)$ and $f(\Omega(f)) \subset \Omega(f)$, the positive orbit of $q$ never enters $B$. Take a neighborhood $V_j$ of $f^{j}(q)$ for each $j = 0, 1, \ldots, k - 2$, such that $V_0, V_1, \ldots, V_{k-2}$ are all disjoint from $B$, that $f(V_j) \subset V_{j+1}$ for $j = 0, 1, \ldots, k - 3$, and that $f(V_{k-2}) \subset U$. Arbitrarily near $p$ and $q$, there are two points $f^{j_i}(x_i)$ and $x_i$ for large $i$. Hence there is a $C^1$ small perturbation $g$ of
f (here g is f composed with a lift. This lift can be even $C^\infty$ close to the identity), supported on $B$, that takes $f^{j_i-1}(x_i)$ onto $x_i$. Note that the g-orbit from $x_i$ to $g^{j_i-1}(x_i)$ are the same as the f-orbit from $x_i$ to $f^{j_i-1}(x_i)$ since the latter intersects $B$ only at the last point $f^{j_i-1}(x_i)$ by the way of choosing $B$. Hence $x_i$ is periodic of g. Since $f = g$ on $V, V_1, \ldots, V_{k-2}$, it follows that $g^{k-1}(x_i) = f^{k-1}(x_i) \in U$. Hence g has a periodic point $g^{k-1}(x_i)$ in U. This proves Theorem A for Case 2, and hence proves the whole Theorem A. Q.E.D.

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REFERENCES


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