THE FEKETE-SZEGÖ PROBLEM
FOR STRONGLY CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Let $K(\beta)$ denote the class of normalized analytic strongly close-to-convex functions of order $\beta \geq 0$, defined in the unit disc $D$ and let $f \in K(\beta)$, with $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, for $z \in D$. Sharp bounds are obtained for $|a_3 - \mu a_2^2|$ when $\mu$ is real.

Introduction

Denote by $S$ the class of normalized analytic univalent functions $f$ defined for $z \in D = \{z : |z| < 1\}$ by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$ 

A classical theorem of Fekete and Szegö [2] states that for $f \in S$ given by (1),

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0 \\ 1 + 2e^{-2\mu/(1-\mu)}, & \text{if } 0 \leq \mu < 1 \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases}$$

This inequality is sharp in the sense that for each $\mu$ there exists a function in $S$ such that equality holds. Recently Pfluger [8] has considered the problem when $\mu$ is complex. In the case of $C^*$, $S^*$ and $K$, the subclasses of convex, starlike and close-to-convex functions respectively, the above inequalities can be improved [5, 6]. In particular for $f \in K$ and given by (1), Keogh and Merkes [5] showed that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 1/3 \\ 1/3 + 4/9\mu, & \text{if } 1/3 \leq \mu \leq 2/3 \\ 1, & \text{if } 2/3 \leq \mu \leq 1 \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases}$$

Again, for each $\mu$, there is a function in $K$ such that equality holds. In this paper we extend this result to the class $K(\beta)$ of strongly close-to-convex functions of order $\beta$ in the sense of Pommerenke [9]. Thus $f \in K(\beta)$ if, and
only if, \( f \), given by (1), is analytic in \( D \) and is such that there exists \( g \in S^* \), satisfying

\[
(2) \quad \left| \arg \frac{zf'(z)}{g(z)} \right| \leq \frac{\pi \beta}{2},
\]

for \( z \in D \) and \( \beta \geq 0 \). Clearly \( K(0) = C \), \( K(1) = K \) and when \( 0 \leq \beta \leq 1 \), \( K(\beta) \) is a subset of \( K \) and hence contains only univalent functions. However in [4], Goodman showed that \( K(\beta) \) can contain functions with unbounded valence for \( \beta > 1 \).

Recently, Koepf [7] has considered the Fekete-Szegö problem for \( K(\beta) \) and obtained sharp results for some particular values of \( \mu \), all of which, with the exception of the case \( \mu = 1 \) and \( \beta \geq 1 \), are contained in the following result.

**Results**

**Theorem.** Let \( f \in K(\beta) \) and be given by (1). Then for \( 0 < \beta < 1 \),

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
1 - \mu + \frac{\beta(2 - 3\mu)(\beta + 2)}{3}, & \text{if } \mu \leq \frac{2\beta}{3(\beta + 1)}, \\
1 - \mu + \frac{2\beta}{3} + \frac{\beta(2 - 3\mu)^2}{3[2 - \beta(2 - 3\mu)]}, & \text{if } \frac{2\beta}{3(\beta + 1)} \leq \mu \leq \frac{2}{3}, \\
\frac{2\beta + 1}{3}, & \text{if } \frac{2}{3} \leq \mu \leq \frac{2(\beta + 2)}{3(\beta + 1)}, \\
\mu - 1 + \frac{\beta(3\mu - 2)(\beta + 2)}{3}, & \text{if } \mu \geq \frac{2(\beta + 2)}{3(\beta + 1)},
\end{cases}
\]

whilst for \( \beta > 1 \), the first two inequalities hold. For each \( \mu \) there are functions in \( K(\beta) \) such that equality holds in all cases.

We shall require the following:

**Lemma 1** ([10, p. 166]). Let \( h \in P \), i.e., let \( h \) be analytic in \( D \) and satisfy \( \text{Re} \, h(z) > 0 \) for \( z \in D \), with \( h(z) = 1 + c_1 z + c_2 z^2 + \cdots \), then

\[
|c_2 - c_1^2| \leq 2 - \frac{|c_1^2|}{2}.
\]

**Lemma 2** ([6, Lemma 3]). Let \( g \in S^* \) with \( g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \), then for \( \mu \) real,

\[
|b_3 - \mu b_2^2| \leq \max\{1, |3 - 4\mu|\}.
\]

We note that Lemma 2 above can easily be extended to the wider class \( S^*(\alpha) \) of strongly starlike functions of order \( \alpha \geq 0 \), i.e., \( g \) analytic and normalized in \( D \) and satisfying

\[
\left| \arg \frac{zg'(z)}{g(z)} \right| \leq \frac{\alpha \pi}{2},
\]

see e.g. [1]. In this case, one obtains the sharp inequality

\[
|b_3 - \mu b_2^2| \leq \max\{\alpha, \alpha^2|3 - 4\mu|\},
\]

for \( \mu \) real.

**Proof of Theorem.** It follows from (2) that we can write

\[
(3) \quad zf'(z) = g(z)h(z)^\beta
\]
for $g \in S^*$ and $h \in P$. Equating coefficients in (3) we obtain $2a_2 = \beta c_1 + b_2$ and $3a_3 = (\beta(\beta - 1)/2)c_1^2 + \beta c_2 + \beta c_1 b_2 + b_3$, so that

$$a_3 - \mu a_2^2 = \frac{1}{3} \left( b_3 - \frac{3}{4} \mu b_2^2 \right) + \frac{\beta}{3} \left( c_2 + \left( \frac{\beta(2 - 3\mu)}{4} - \frac{1}{2} \right) c_1^2 \right) + \beta \left( \frac{1}{3} - \frac{\mu}{2} \right) c_1 b_2.$$  

(4)

We consider first the case $(2\beta)/(3(\beta + 1)) \leq \mu \leq 2/3$. Equation (4) gives

$$|a_3 - \mu a_2^2| \leq \left| b_3 - \frac{3}{4} \mu b_2^2 \right| + \frac{\beta}{3} \left| c_2 - \frac{1}{2} c_1^2 \right| + \frac{\beta^2(2 - 3\mu)}{12} |c_1|^2 + \beta \left( \frac{1}{3} - \frac{\mu}{2} \right) |c_1||b_2|, \leq 1 - \mu + \frac{\beta}{3} \left( 2 - \frac{1}{2} c_1^2 \right) + \frac{\beta^2(2 - 3\mu)}{12} |c_1|^2 + \frac{\beta(2 - 3\mu)}{3} |c_1|,$$

$$= \Phi(x) \text{ say, with } x = |c_1|,$$

where we have used Lemmas 1 and 2 and the fact that $|b_2| \leq 2$ for $g \in S^*$. An elementary argument shows that the function $\Phi$ attains a maximum at $x_0 = 2(2 - 3\mu)/(2 - 2(\beta - 2\mu))$, and so

$$|a_3 - \mu a_2^2| \leq \Phi(x_0),$$

which proves the theorem if $\mu \leq 2/3$ and $\beta > 0$. Choosing

$$c_1 = \frac{2(2 - 3\mu)}{2 - 2(\beta - 2\mu)}, \quad c_2 = 2, \quad b_2 = 2, \quad \text{and} \quad b_3 = 3,$$

in (4) shows that the result is sharp. We note that $|c_1| \leq 2$, that is, $\mu \geq 2\beta/(3(\beta + 1))$.

Next consider the case $\mu \leq (2\beta)/(3(\beta + 1))$. Then

$$|a_3 - \mu a_2^2| \leq \left| a_3 - \frac{2\beta}{3(\beta + 1)} a_2^2 \right| + \left( \frac{2\beta}{3(\beta + 1)} - \mu \right) |a_2|^2,$$

$$\leq 1 + \frac{2\beta}{3} + \left( \frac{2\beta}{3(\beta + 1)} - \mu \right) (\beta + 1)^2 = 1 - \mu + \frac{\beta(2 - 3\mu)(\beta + 2)}{3},$$

for $\beta \geq 0$, where we have used the result already proved in the case $\mu = 2\beta/(3(\beta + 1))$, and the fact that for $f \in K(\beta)$, the inequality $|a_2| \leq \beta + 1$ holds [3]. Equality is attained on choosing $\lambda = 0$, $p_1 = p_2 = b_2 = 2$, and $b_3 = 3$.

Suppose now that $2/3 \leq \mu \leq (2(\beta + 2))/(3(\beta + 1))$. Since $g \in S^*$ we can write $z g'(z) = g(z)p(z)$ for $p \in P$, with $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, and so equating coefficients we have that $b_2 = p_1$ and $2b_3 = p_1^2 + p_2$.

We deal first with the case $\mu = 2(\beta + 2)/(3(\beta + 1))$. Thus (4) gives

$$a_3 - \frac{2(\beta + 2)}{3(\beta + 1)} a_2^2 = \frac{1}{6} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{\beta}{3} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{\beta - 1}{12(\beta + 1)} p_1^2,$$

$$- \frac{\beta^2 c_1^2}{6(\beta + 1)} - \frac{\beta p_1 c_1}{3(\beta + 1)},$$
and so if $\beta \leq 1$,
\[
\left| a_3 - \frac{2(\beta + 2)}{3(\beta + 1)} a_2^2 \right| \leq \frac{1}{6} \left| p_2 - \frac{p_1^2}{2} \right| + \frac{\beta}{3} \left| c_2 - \frac{c_1^2}{2} \right| + \frac{(1 - \beta)}{12(\beta + 1)} |p_1^2|
\]
\[
+ \frac{\beta^2 |c_2^2|}{6(\beta + 1)} + \frac{\beta |p_1 c_1|}{3(\beta + 1)},
\]
\[
\leq \frac{1}{6} \left( 2 - \frac{|p_1^2|}{2} \right) + \frac{\beta}{3} \left( 2 - \frac{|c_1^2|}{2} \right) + \frac{1 - \beta}{12(\beta + 1)} |p_1^2|
\]
\[
+ \frac{\beta^2 |c_2^2|}{6(\beta + 1)} + \frac{\beta |p_1 c_1|}{3(\beta + 1)},
\]
\[
= \frac{2\beta + 1}{3} - \frac{\beta}{6(\beta + 1)} (|c_1| - |p_1|)^2,
\]
\[
\leq \frac{2\beta + 1}{3},
\]
where we have used Lemma 1.

Now write
\[
a_3 - \mu a_2^2 = \frac{(\beta + 1)(3\mu - 2)}{2} \left( a_3 - \frac{2(\beta + 2)}{3(\beta + 1)} a_2^2 \right)
\]
\[
+ \frac{3(\beta + 1)}{2} \left( \frac{2(\beta + 2)}{3(\beta + 1)} - \mu \right) (a_3 - \frac{2}{3} a_2^2),
\]
and the result follows at once on using the theorem already proved in the cases $\mu = 2/3$ and $\mu = 2(\beta + 2)/(3(\beta + 1))$ for $\beta \leq 1$. Equality is attained when $f$ is given by

\[
f'(z) = \frac{(1 + z^2)^{\beta}}{(1 - z^2)^{\beta+1}}.
\]

We finally assume that $\mu \geq (2(\beta + 2))/(3(\beta + 1))$. Write
\[
a_3 - \mu a_2^2 = \left( a_3 - \frac{2(\beta + 2)}{3(\beta + 1)} a_2^2 \right) + \left( \frac{2(\beta + 2)}{3(\beta + 1)} - \mu \right) a_2^2,
\]
and the result follows at once on using the theorem already proved for $\mu = 2(\beta + 2)/(3(\beta + 1))$ in the case $\beta \leq 1$ and the inequality $|a_2| \leq \beta + 1$, which was proved in [3]. Equality is attained in this last case on choosing $c_1 = b_2 = 2i$, $c_2 = -2$ and $b_3 = -3$ in (4).

We remark that the methods used in [5, 6], together with equation (4), suggest that in order to obtain sharp results for $\beta > 1$ and $\mu > 2/3$, an extension to the “area principle” may be required. Since $K(\beta)$ contains functions of unbounded valence for $\beta > 1$ establishing sharp estimates in this case may require deeper methods.

References


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