A REMARK ON BOURGAIN ALGEBRAS ON THE DISK

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Abstract. It is shown that the Bourgain algebra $X_b$ of the space $X = H^\infty$ considered as a subalgebra of $\mathcal{U} = \text{alg}(H^\infty, \overline{H^\infty})$ is $\mathcal{U}^0(\mathbb{B}) + UC(\mathbb{D})$ where $UC(\mathbb{D})$ is the algebra of uniformly continuous functions on the open unit disk $\mathbb{D}$. This uses and extends a recent result of Cima-Janson-Yale on the Bourgain algebra of $H^\infty$ on $\partial \mathbb{D}$. Further, $(X_b)_b = X_b$.

In [3] Cima and Timoney introduce the concept of Bourgain algebra $X_b$ of a linear subspace $X$ of a Banach algebra $A$ and show that if $X$ itself is an algebra, then $X \subseteq X_b$. In [2] Cima, Janson, and Yale describe the Bourgain algebra of $H^\infty(\partial \mathbb{D})$. Since then there has been further study of Bourgain algebras on the bitorus and the polydisk [8].

We look at $X = H^\infty$ as a subalgebra of $\mathcal{U}$ where $\mathcal{U} = \text{alg}(H^\infty, \overline{H^\infty}) \subseteq \mathcal{L}^\infty(\mathbb{D})$. We recall that $\mathcal{U}$ may be identified with $C(\mathcal{M})$ where $\mathcal{M}$ is the maximal ideal space of $H^\infty$. $UC(\mathbb{D})$ denotes the algebra of uniformly continuous functions on $\mathbb{D}$. It is known that $H^\infty + UC(\mathbb{D}) = H^\infty[z]$, the subalgebra of $\mathcal{L}^\infty(\mathbb{D})$ generated by $H^\infty$ and $z$ (see [1, p. 721]). The only norms we use are the supnorms and essential supnorms; $\|\varphi\|_{\partial \mathbb{D}} = \text{ess sup}\{|\varphi(z)|, z \in \partial \mathbb{D}\}$ whereas $\|\varphi\|_{\mathbb{D}} = \text{sup}\{|\varphi(z)|, z \in \mathbb{D}\}$. An infinite sequence $\{w_n\}_{n=1}^{\infty} \subseteq \mathbb{D}$ is called thin if

$$\prod_{m=1}^{\infty} \left| \frac{w_n - w_m}{1 - \overline{w_n}w_m} \right| \to 1 \quad \text{as} \quad n \to \infty.$$ 

A thin Blaschke product $b$ is a function on $\mathbb{D}$ of the form

$$b(z) = \prod_{n=1}^{\infty} \frac{|w_n|}{w_n} \frac{w_n - z}{1 - \overline{w_n}z},$$

where $\{w_n\}_1$ is a thin sequence. For a Blaschke product $b$ with zeros $\{z_n\}$ the zero set $Z(b)$ in $\mathcal{M} = \text{cl}\{z_n, \ n \geq 1\} \setminus \{z_n, \ n \geq 1\}$ is the zero set of $b$ in $\mathcal{M} \setminus \mathbb{D}$.

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Our main result shows that analogous to (1) we have: \((H^\infty(\mathbb{D}))_b = H^\infty + UC(\mathbb{D})\). This is not particularly surprising, but the investigation does raise some interesting questions about subalgebras of \(L^\infty(\mathbb{D})\).

**Lemma 1.** If \(X = H^\infty(\mathbb{D})\) and \(A = \mathcal{U}\), then we have:

(i) \(H^\infty + UC(\mathbb{D}) \subseteq X_b\);

(ii) if \(b\) is an infinite Blaschke product, then \(\bar{b} \notin X_b\).

**Proof.** The proof of (i) is an easy consequence of the description of \((H^\infty(\partial\mathbb{D}))_b\) and (ii) uses the fact that if \(\phi\) denotes the harmonic extension of \(\varphi \in L^\infty(\partial\mathbb{D})\) to \(\mathbb{D}\) via the Poisson kernel, then the map \(\varphi \mapsto \phi\) is asymptotically multiplicative on \(H^\infty + C(\partial\mathbb{D})\). (See [4, Lemma 6.44].) We sketch the details for completeness.

To prove (i) we only need to prove that \(\bar{z} \in X_b\). So assume that \(f_n \in H^\infty\), \(f_n \to 0\) weakly and let \(\varphi_n(z) = (f_n(z) - f(0))/z\). Then \(\varphi_n \in H^\infty\) and \(\varphi_n \to 0\) uniformly on compacta, and \(\|\varphi_n\| \leq c\), for some \(c > 0\). Hence \(f_n \to 0\), \(\varphi_n \to 0\), both uniformly on compacta, and hence \(\|\varphi_n\| \leq c\). Thus \(\left|\varphi_n(z)\right| \to 0\) for \(z \in \partial\mathbb{D}\). This implies that \(\|\varphi_n\| \to 0\). However, \(f_n(z) - \varphi_n(z) \in H^\infty\) and hence \(\|f_n(z) - \varphi_n(z)\| \to 0\). Clearly, \(\|\varphi_n(z)\| \to 0\). This shows that \(\|\varphi_n(z)\| \to 0\) and hence \(\bar{z} \in X_b\).

(ii) If \(b\) is an infinite Blaschke product such that \(\bar{b} \in X_b\), then clearly \(\bar{b}_{|\partial\mathbb{D}} \in (H^\infty(\partial\mathbb{D}))_b\), but \(\bar{b}_{|\partial\mathbb{D}} \neq \varphi + \psi\), where \(\varphi \in H^\infty\) and \(\psi \in C(\partial\mathbb{D})\). In particular \(1 \neq b\varphi + \psi\) a.e. on \(\partial\mathbb{D}\). Using [4, Lemma 6.44] we have \(1 \approx \varphi + \psi\) a.e. on the annulus \(\{z, \rho < |z| < 1\}\). This is clearly impossible since \(\{z, \rho < |z| < 1\} \cap \{\text{zeros of } b\} \neq 0\).

**Remark 1.** In the context of Hankel operators and their compactness a natural analogue of \(H^\infty + C(\partial\mathbb{D})\) on the disk is \(H^\infty + COP\), see [1, §6], and on first thought \(H^\infty + COP\) may be considered a natural candidate for \((H^\infty(\partial\mathbb{D}))_b\). However it follows as an immediate consequence of Lemma 1 that \(COP \not\subset (H^\infty(\partial\mathbb{D}))_b\), because \(COP\) contains \(\bar{b}\) whenever \(b\) is a Blaschke product in the little Bloch space \(B_0\) and \(B_0\) is known to contain infinite Blaschke products [5, p. 442]. K. Stroethoff and K. Yale have studied \(H^\infty(\mathbb{D})_b\) as a subalgebra of \(L^\infty(\mathbb{D})\) and found a function \(f \in L^\infty(\mathbb{D})\) such that the Hankel operator \(H_f\) is compact but \(f \notin H^\infty(\mathbb{D})_b\). They also show that in the larger context of \(L^\infty(\mathbb{D})\), \(H^\infty(\mathbb{D})_b \supseteq H^\infty + UC(\mathbb{D})\) (private communication).

As the next proposition shows the proper analogue of \(H^\infty + C(\partial\mathbb{D})\) in the context of Bourgain algebras within \(\mathcal{U}\) is \(H^\infty + UC(\mathbb{D})\).

**Proposition 1.** \((H^\infty(\mathbb{D}))_b = H^\infty + UC(\mathbb{D})\).

**Proof.** In view of Lemma 1 we only need to prove that \((H^\infty(\mathbb{D}))_b \subseteq H^\infty + UC(\mathbb{D})\). Suppose \(f \in \mathcal{U}\) and \(f \in (H^\infty(\mathbb{D}))_b\). Then \(f\) has nontangential limits a.e. and we denote this function by \(f_{|\partial\mathbb{D}}\). Clearly \(\varphi = f - (f_{|\partial\mathbb{D}}) \in C(\mathbb{D})\) and \(\varphi_{|\partial\mathbb{D}} = 0\) a.e.

**Claim 1.** \(\varphi \in (H^\infty(\mathbb{D}))_b\).
To prove the claim note that by hypothesis, \( f \in (H^\infty(D))_b \) and hence \( f|_{\partial D} \in H^\infty + C(\partial D) \) [2, Theorem 1]. It follows that \( (f|_{\partial D})^\sim \in H^\infty + U(C(D)) \). By Lemma 1, \( (f|_{\partial D})^\sim \in (H^\infty(D))_b \) and this completes proof of the claim.

Claim 2. Given \( \varepsilon > 0 \) \( \exists r(0 < r < 1) \) such that \( |\psi(z)| < \varepsilon \) for \( z \in \{0 < |z| < 1\} \).

If not \( \exists \{r_n\}, 0 < r_n < 1, r_n \to 1 \) and \( \{z_n\} \subseteq D \) such that \( |z_n| > r_n \) and \( |\psi(z_n)| \geq \delta \). Choosing a subsequence if necessary, we may assume that \( \{z_n\} \) is an interpolating sequence. As in [2, p. 123] we have a sequence \( \{f_n\} \subseteq H^\infty \), such that \( f_n(z_m) = \delta_{nm} \) and \( f_n \overset{wk}{\to} 0 \). By Claim 1, \( \exists \{\phi_n\} \subseteq H^\infty(D) \) such that \( ||\psi f_n - \phi_n||_D < \varepsilon_n \) where \( \varepsilon_n \to 0 \). In particular, \( ||\phi_n||_D < \varepsilon_n \) and \( \phi_n \) being in \( H^\infty \) this implies \( ||\phi_n||_D < \varepsilon_n \). Thus \( ||\psi f_n|| < 2\varepsilon_n \to 0 \), which is clearly a contradiction since \( ||\psi f_n(z_n)|| = ||\psi(z_n)|| \geq \delta \) for all \( n \). This proves Claim 2 and we conclude that \( \psi \in C_0(D) \subseteq U(C(D)) \) and hence \( f = (f|_{\partial D})^\sim + \psi \in H^\infty + U(C(D)) \).

Proposition 2. \( (H^\infty + U(C(D)))_b = H^\infty + U(C(D)) \).

Proof. It is sufficient to show that \( (H^\infty + U(C(D)))_b \subseteq H^\infty + U(C(D)) \). Let \( f \in (H^\infty + U(C(D)))_b \) and write \( g = f|_{\partial D} \). This is well defined since by assumption \( f \in Z = C(\mathcal{M}) \). Clearly \( g \in (H^\infty + C(\partial D))_b \) and hence \( g \in H^\infty + C(\partial D) \). See [6]. In particular, \( \hat{g} \equiv \) the harmonic extension of \( g \) to \( D \) belongs to \( H^\infty + U(C(D)) \subseteq (H^\infty + U(C(D)))_b \) and hence \( \psi = f - \hat{g} \in (H^\infty + U(C(D))) \). We claim that \( \psi \in C_0(D) \). If not, (dropping to a subsequence) we may choose \( \{z_n\} \subseteq D \) such that \( \{z_n\} \) is a thin sequence satisfying \( |\psi(z_n)| \geq \delta \) for some \( \delta > 0 \). Let \( b \) be the (thin) Blaschke product with zeros \( \{z_n\} \). Following a construction of Izuchi's (the ideas behind which are outlined in [7]) we can find \( \{y_n\} \subseteq Z(b) \) and \( \{f_n\} \subseteq H^\infty \) such that

1. \( f_n \overset{wk}{\to} 0 \)
2. \( |f_n(y_n)| > 1 - \varepsilon_n \) where \( \varepsilon_n \to 0 \).

Since we have not seen it in print, we provide a very brief sketch for completeness. Suppose \( x \in \mathcal{M} \) and \( \mu_x \) is its unique representing measure supported on the Shilov boundary of \( \mathcal{M} \). By a well-known but unpublished result of Hoffman's, whenever \( x_1 \) and \( x_2 \) are in the zero set \( Z(b) \) of a thin Blaschke product \( b \), \( \mu_x \), and \( \mu_{x_2} \) have disjoint supports and hence starting with an arbitrary sequence \( \{x_n\} \) in \( Z(b) \) with a cluster point \( x_0 \) (\( \in Z(b) \)) we have \( \mu_{x_0} \cap \text{spt} \mu_{x_n} = \phi \) for all \( n \). The \( f_n \) s are now built inductively using the fact that \( \text{spt} \mu_{x_n} (n \geq 0) \) is a weak peak set for \( H^\infty \) (unpublished manuscript).

Since \( \psi \in (H^\infty + U(C(D))_b \exists g_n \in H^\infty \) and \( \phi_n \in U(C(D)) \) such that \( ||\psi f_n - g_n - \phi_n|| < \varepsilon_n \to 0 \). As \( \psi|_{\partial \mathcal{M}}(L^\infty(D)) = 0 \), we have \( ||g_n + \phi_n||_{L^\infty(D)} \leq \varepsilon_n \) and replacing \( \phi_n \) by the harmonic extension \( \hat{\phi}_n \) of \( \phi_n \) we have

\[
(*) \quad ||g_n + \hat{\phi}_n|| \leq \varepsilon_n.
\]

Clearly \( \phi_n - \hat{\phi}_n \) vanishes on \( \mathcal{M} \) and for any \( m \in \mathcal{M} \setminus D \), \( g_n(m) + \phi_n(m) = g_n(m) + \hat{\phi}_n(m) \) and hence \( ||\psi f_n - g_n - \phi_n||_{Z(b)} \leq \varepsilon_n \). So \( \|g + \hat{\phi}_n\|_{Z(b)} \geq ||\psi f_n\|_{Z(b)} - ||\psi f_n - g_n - \phi_n\|_{Z(b)} \geq ||\psi f_n\|_{Z(b)} - \varepsilon_n \).

However, \( \psi \in C(\mathcal{M}) \) and \( ||\psi(z_n)|| \geq \delta \), so \( ||\psi(m)|| \geq \delta \) for all \( m \in Z(b) \) and hence \( ||\psi f_n\|_{Z(b)} \geq \delta |f_n(y_n)| \geq (1 - \varepsilon_n) \delta \). It follows that \( ||g + \hat{\phi}_n||_{Z(b)} \geq \delta / 2 \).
for $n$ sufficiently large. This clearly contradicts (*) and proves the claim that $\psi \in C_0(\overline{D})$. In particular $f = \hat{g} + \psi \in H^\infty + UC(D)$.

**Remark 2.** In [6] Gorkin and Izuchi showed that the only Douglas algebra $B$ for which $B_B = L^\infty(\partial D)$ is $L^\infty(\partial D)$ and that if $A$ and $B$ are Douglas algebras with $A \subseteq B$ then $A_B \subseteq B_B$. Our work indicates that both of these analogues cannot be valid on the disk. Dechao Zheng has some further results on subalgebras $A$ of $\mathcal{U}$ for which $A_B = \mathcal{U}$. However, the proofs are fairly involved and will be published elsewhere.

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**References**


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