

ZAPPING SMALL FILTERS

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ABSTRACT. We show two methods for diagonalizing filters of the form F_σ , first without adding an unbounded real, the other while preserving P -points; the interest lies in an attempt at destroying maximal almost disjoint families with least damage.

1. INTRODUCTION

Our aim is to investigate how to diagonalize, or zap filters on the natural numbers with the least harm done. Indeed, a natural approach to several open questions on “cardinal invariants of the continuum” such as the consistency of the inequalities $\mathfrak{d} < \mathfrak{a}$, $\mathfrak{i} < \mathfrak{a}$, $\mathfrak{u} < \mathfrak{a}$ (see [6]), is to iteratively destroy maximal almost disjoint families while preserving the appropriate objects. Obviously, the only way (up to equivalence) to destroy a maximal almost disjoint family \mathcal{A} is to diagonalize the filter $\mathcal{F}(\mathcal{A})$ generated by the collection $\{A^c : A \in \mathcal{A}\}$.

We first develop a necessary condition for filters to be diagonalized in a forcing extension without adding an unbounded real, and then we verify that filters of the form $\mathcal{F}(\mathcal{A})$ do satisfy the condition. We only succeed however, in diagonalizing filters of the form F_σ without adding an unbounded real. In the following section, we indicate how to diagonalize F_σ while preserving P -points from the ground model.

This of course allows us also to diagonalize any filter that is included in an F_σ filter while either preserving P -points or not adding an unbounded real. We finally show in the last section how to build in ZFC maximal almost disjoint families \mathcal{A} for which $\mathcal{F}(\mathcal{A})$ is included in an F_σ filter; however, the Continuum Hypothesis allows us to build an \mathcal{A} such that $\mathcal{F}(\mathcal{A})$ is not included in any F_σ filter and hence we probably have provided only a very slim partial result on the consistency of either $\mathfrak{d} < \mathfrak{a}$ or $\mathfrak{u} < \mathfrak{a}$.

We now briefly review some terminology.

We write ω for the set of natural numbers, $[\omega]^\omega$ for the set of its infinite subsets, and ω^ω for the set of functions on the natural numbers. $X \subseteq^* Y$ means that $X \setminus Y$ is finite, and $X =^* Y$ means that $X \setminus Y \cup Y \setminus X$ is finite. A natural number k is often identified with the set of its predecessors $\{0, \dots, k-1\}$.

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A filter \mathcal{F} is a family of subsets of ω closed under supersets and finite intersections; we assume that all our filters consist only of infinite sets and contain the Fréchet filter $\mathcal{F}r$ of all cofinite sets. More generally, a collection of infinite sets is said to have the finite intersection property if the intersection of any finite subcollection is infinite. A set $X \in [\omega]^\omega$ is said to diagonalize, or to zap, a filter \mathcal{F} , if $X \subseteq^* Y$ for each $Y \in \mathcal{F}$. A family $\mathcal{A} \subseteq [\omega]^\omega$ is almost disjoint if the pairwise intersection of (distinct) members of \mathcal{A} is finite. We will also identify subsets of ω with their characteristic functions in 2^ω , the latter space being equipped with the product topology of the discrete topology on $2 = \{0, 1\}$.

Given two models $\mathcal{M} \subseteq \mathcal{N}$, we say that $f \in (\omega^\omega)^\mathcal{N}$ is unbounded if

$$(\forall g \in (\omega^\omega)^\mathcal{M})(\exists^\infty n)[f(n) \geq g(n)]$$

and dominating if

$$(\forall g \in (\omega^\omega)^\mathcal{M})(\forall^\infty n)[f(n) \geq g(n)].$$

Here $\exists^\infty n$ means “there are infinitely many” and $\forall^\infty n$ means “for all but finitely many n .” An ultrafilter $\mathcal{U} \in \mathcal{M}$ is said to be preserved if, in \mathcal{N} , $\overline{\mathcal{U}} = \{X \subseteq \omega : (\exists Y \in \mathcal{U}) Y \subseteq X\}$ is an ultrafilter.

2. A NECESSARY CONDITION

We attack the problem of diagonalizing a filter \mathcal{F} without adding an unbounded real. It is clear first of all that the enumerating functions of sets in \mathcal{F} should form a bounded family, which turns out by a result of Talagrand [5] to be equivalent to \mathcal{F} being meager. Of course, diagonalizing \mathcal{F} also zaps the filters $f(\mathcal{F}) = \{X : f^{-1}(X) \in \mathcal{F}\}$ and hence the same should be true for those filters; this means that \mathcal{F} is required to be “completely meager” (see [4]). Consider further the following situation: think of \mathcal{F} as a filter on $\omega \times \omega$ and suppose it contains the set above the graph of each function in ω^ω ; then diagonalizing \mathcal{F} would actually add a dominating real.

Putting that together we get:

Proposition 2.1. *If \mathcal{F} can be diagonalized in a forcing extension without adding an unbounded real, then it is necessary that for each family $\langle X_i : i \in \omega \rangle$ such that:*

1. *Each X_i is incompatible with \mathcal{F} , (that is $X_i^c \in \mathcal{F}$)*
2. *$\bigcup_{i \in \omega} X_i \in \mathcal{F}$.*

Then there exist $h \in \omega^\omega$ and a sequence $n_0 < n_1 < \dots$ such that:

$$(\forall Y \in \mathcal{F})(\forall^\infty k) \left[Y \cap \bigcup_{[n_k, n_{k+1})} X_i \cap h(i) \neq \emptyset \right].$$

Proof. Suppose that some $Z \in [\omega]^\omega$ diagonalizes \mathcal{F} in some forcing extension, and suppose that the condition fails for the family $\langle X_i : i \in \omega \rangle$ as above. We produce an unbounded real from Z . We consider two cases:

Case 1. (For all sequences $n_0 < n_1 < \dots$)($\exists Y \in \mathcal{F})(\exists^\infty k)[Y \cap \bigcup_{i \in [n_k, n_{k+1})} X_i = \emptyset$].

This case corresponds to the situation when $h(\mathcal{F})$ is not meager, where $h(n) =$ the smallest i such that $n \in X_i$.

Define $\bar{Z} = \{i: X_i \cap Z \neq \emptyset\}$, which is infinite by the hypothesis on the family $\langle X_i: i \in \omega \rangle$, and put $f_Z(n) = n$ th element of \bar{Z} . We claim that f_Z is unbounded. Indeed fix $g \in (\omega^\omega)^\vee$, and define a sequence of natural numbers such that $n_{k+1} > g(n_k)$ for each k . By assumption we can find some $Y \in \mathcal{F}$ such that

$$(\exists^\infty k) \left[Y \cap \bigcup_{i \in [n_k, n_{k+1})} X_i = \emptyset \right];$$

hence for each such k , we get:

$$f_Z(n_k) \geq n_{k+1} \geq g(n_k)$$

as desired.

Case 2. $(\exists n_0 < n_1 < \dots)(\forall Y \in \mathcal{F})(\forall^\infty k)[Y \cap \bigcup_{i \in [n_k, n_{k+1})} X_i \neq \emptyset]$.

Hence by assumption, it must be that:

$$(\forall h \in \omega^\omega)(\exists Y \in \mathcal{F})(\exists^\infty k) \left[Y \cap \bigcup_{i \in [n_k, n_{k+1})} X_i \cap h(i) = \emptyset \right].$$

Now define

$$f_Z(k) = \min \left\{ Z \cap \bigcup_{i \in [n_k, n_{k+1})} X_i \right\}.$$

Clearly, f_Z is an unbounded function. \square

The first nontrivial example of filters satisfying the condition are filters of the form $\mathcal{F}(\mathcal{A})$ where \mathcal{A} is a maximal almost disjoint family; they actually satisfy a slightly stronger condition.

Proposition 2.2. *Let \mathcal{A} be a maximal almost disjoint family and $\mathcal{F} = \mathcal{F}(\mathcal{A})$. Then for all families $\langle X_i: i \in \omega \rangle$ such that:*

- (1) *Each X_i is incompatible with \mathcal{F} ,*
- (2) *$\bigcup_{i \in \omega} X_i$ is compatible with \mathcal{F} .*

Then there are $h \in \omega^\omega$ and a sequence $n_0 < n_1 < \dots$ such that:

$$(\forall Y \in \mathcal{F})(\forall^\infty k) \left[Y \cap \bigcup_{i \in [n_k, n_{k+1})} X_i \cap h(i) \neq \emptyset \right].$$

Lemma 2.3. *There are infinitely many $A \in \mathcal{A}$ such that $(\forall n)[A \cap \bigcup_{i > n} X_i \neq \emptyset]$.*

Proof. Assume that there are only finitely many such sets A , say $\{A_i: i < l\}$. By induction, try to build $C = \{c_n: n \in \omega\}$ such that

$$c_n \in \bigcup_{i > n} X_i \setminus \bigcup_{i < l} A_i.$$

If we succeed and C is infinite, then the family $\mathcal{A} \cup \{C\}$ would contradict the maximality of \mathcal{A} since it is almost disjoint. Hence C must be finite and

$$\bigcup_{i > n} X_i \subseteq^* \bigcup_{i < l} A_i \text{ for some } n.$$

This means that the set $(\bigcup_{i > n} X_i)^c$ belongs to \mathcal{F} and hence that $\bigcup_{i < n} X_i$ is compatible with \mathcal{F} , a contradiction. \square

Now to prove the proposition, fix a family $\{A_i: i \in \omega\}$ as in the lemma, and define a function $h \in \omega^\omega$ and a sequence $n_0 < n_1 < \dots$ such that

$$(\forall j \leq k) \bigcup_{i \in [n_k, n_{k+1})} X_i \cap h(i) \cap A_j \neq \emptyset.$$

We claim that this works. Indeed, suppose $Y \in \mathcal{F}$, say $Y =^* C_0^c \cap \dots \cap C_r^c$, where $C_i \in \mathcal{A}$, and that

$$(\exists^\infty k) \left[Y \cap \bigcup_{i \in [n_k, n_{k+1})} X_i \cap h(i) = \emptyset \right].$$

Then

$$(\exists^\infty k) \bigcup_{i \in [n_k, n_{k+1})} X_i \cap h(i) \subseteq C_0 \cup \dots \cup C_r.$$

But this means that $A_j \cap \bigcup_{i \leq r} C_i$ is infinite for each j , and hence is a contradiction. \square

The second example consists of the filters of the form F_σ , that is filters that can be written as a countable union of closed sets. The Fréchet filter is such a filter, since $\mathcal{F}r = \bigcup_{i \in \omega} C_i$ where $C_i = \{X \subseteq \omega: \text{the size of } X^c \text{ is at most } i\}$ is a closed set.

The following, taken from [4], shows that all the information regarding a F_σ filter is contained in a closed set.

Lemma 2.4. *If \mathcal{F} is an F_σ filter, then there is a closed set C with the finite intersection property which generates \mathcal{F} (i.e., $(\forall X \in \mathcal{F})(\exists Y \in C)[Y \subseteq^* X]$).*

Proof. If $\mathcal{F} = \bigcup_{i \in \omega} C_i$, then let $C = \{X \cup i: X \in C_i\}$. \square

Proposition 2.5. *Let \mathcal{F} be an F_σ filter and $\langle X_i: i \in \omega \rangle$ be a family such that:*

- (1) *Each X_i is incompatible with \mathcal{F} ,*
- (2) *$\bigcup_{i \in \omega} X_i$ is compatible with \mathcal{F} .*

Then there are $h \in \omega^\omega$ and a sequence $n_0 < n_1 < \dots$ such that:

$$(\forall Y \in \mathcal{F})(\forall^\infty k) \left[Y \cap \bigcup_{i \in [n_k, n_{k+1})} X_i \cap h(i) \neq \emptyset \right].$$

Proof. Fix a closed set C which generates \mathcal{F} as in the lemma. Proceeding by induction, suppose that we have already defined n_k and $h \upharpoonright n_k$. Let $n = h(n_k - 1)$ and $X = \bigcup_{i \geq n_k} X_i$, which is compatible with \mathcal{F} by assumption. Then $C(X) = \{Y \cap X: Y \in C\}$ is a closed set with the finite intersection property, and hence we can find $m > n$ such that $Z \cap [n, m) \neq \emptyset$ for each $Z \in C(X)$. Now choose n_{k+1} large enough so that

$$(\forall Y \in C) Y \cap [n, m) \cap \bigcup_{i \in [n_k, n_{k+1})} X_i \neq \emptyset$$

and put $h \upharpoonright [n_k, n_{k+1}) = m$. \square

In the next section we show how to diagonalize F_σ filters without adding an unbounded real. It is more difficult to find a reasonable necessary condition for a filter to be diagonalizable in a forcing extension while preserving some

ultrafilters. However, since the enumerating functions of sets in an ultrafilter always form an unbounded family, an obvious requirement is that we do not add a dominating real. It should be noticed that analytic filters do not in general satisfy the above necessary condition; indeed, diagonalizing an analytic filter will in general add a dominating real (see [4]).

3. F_σ FILTERS AND UNBOUNDED REALS

We now show that we can diagonalize F_σ filters without adding an unbounded real.

Theorem 3.1. *It is possible to diagonalize any F_σ filter in a forcing extension without adding an unbounded real.*

First, we look in more detail at the structure of those filters.

Lemma 3.2. *Let F be an F_σ filter. Then there are a sequence $n_0 < n_1 < \dots$ and sets $a_0^k, \dots, a_{r_k-1}^k \subseteq [n_k, n_{k+1})$ for each k such that:*

- (1) $(\forall x \in [r_k]^k) \bigcap_{i \in x} a_i^k \neq \emptyset,$
- (2) $(\forall X \in \mathcal{F})(\exists f \in \prod_{k \in \omega} r_k) [\bigcup \{a_{f(k)}^k : k \in \omega\} \subseteq^* X].$

i.e., \mathcal{F} is included in the (F_σ) filter generated by the sets $\{X : (\forall k)(\exists i < r_k) a_i^k \subseteq X\}$.

Proof. As in Lemma 2.4, let C be a closed set with the finite intersection property which generates \mathcal{F} . Given n_k , choose n_{k+1} large enough so that

$$(\forall X_1, \dots, X_k \in C)[X_1 \cap \dots \cap X_k \cap [n_k, n_{k+1}) \neq \emptyset].$$

Now list $\{X \cap [n_k, n_{k+1}) : X \in C\}$ as $a_0^k, \dots, a_{r_k-1}^k$. This completes the proof. \square

According to the lemma, we have reduced the problem of zapping F_σ filters to the following problem:

Given $r_k \in \omega$, for $k \in \omega$, forcing $x_k \in [r_k]^k, k \in \omega$, such that

$$\left(\forall f \in \prod_{k \in \omega} r_k \right)^V (\forall^\infty k)[f(k) \in x_k],$$

and without adding an unbounded real. Indeed, given the family $\langle x_k : k \in \omega \rangle$ as above, the set $Z = \{z_k : k \in \omega\}$ where $z_k \in \bigcap_{i \in x_k} a_i^k (\neq \emptyset)$, will diagonalize the filter \mathcal{F} .

Although some specific cases can be dealt with simply by adding a random real (when r_k is not too large compared to k), the essential idea of the general case is due to Shelah [1] (i.e., forcing with perfect trees that are finitely but “profusely” branching does not add an unbounded real).

First let $s_k = \binom{r_k}{k}$ and list $[r_k]^k = \{x_0^k, \dots, x_{s_k-1}^k\}$. Define

$$\mathbb{P} = \{T : T \text{ is a perfect tree of } \bigcup_{n \in \omega} \prod_{k < n} s_k \text{ such that}$$

$$(\forall n)(\exists \bar{m})(\forall m \geq \bar{m})(\forall a_1, \dots, a_n \in r_m)(\forall s \in \omega^m \cap T) \\ |\{t \in \text{succ}_T(s) : \{a_1, \dots, a_n\} \subseteq x_{t(m)}^m\}| \geq n\}$$

with ordering being inclusion.

As usual, for $s \in T$, T_s will denote the collection of nodes from T that are comparable with s .

Claim 3.3. If G is \mathbb{P} -generic and $g = \bigcap_{T \in G} T$, then

$$\left(\forall f \in \prod_{k \in \omega} r_k \right)^V (\forall^\infty k)[f(k) \in x_{g(k)}^k]$$

as desired, i.e., the desired x_k is $x_{g(k)}^k$.

Proof. Given $f \in \prod_{k \in \omega} r_k$ which belongs to the ground model V , it suffices to verify that

$$\Delta(f) = \{T \in \mathbb{P} : (\exists \bar{m})(\forall m \geq \bar{m})(\forall s \in \omega^{m+1} \cap T)f(m) \in x_{s(m)}^m\}$$

is dense. Now given $T \in \mathbb{P}$, choose \bar{m} such that

$$\begin{aligned} &(\forall m \geq \bar{m})(\forall a_1, a_2 \in r_m)(\forall s \in \omega^m \cap T) \\ &|\{t \in \text{succ}_T(s) : \{a_1, a_2\} \subseteq x_{t(m)}^m\}| \geq 2; \end{aligned}$$

then $T' = T \cap \{s : (\forall m \geq \bar{m})f(m) \in x_{s(m)}^m\} \in \Delta(f)$ as desired. \square

We only need to show that \mathbb{P} adds no unbounded real.

Definition 3.4. For $k \in \omega$, we define:

$$\begin{aligned} T' \leq_k T \quad \text{if } &(\forall m)(\forall s \in \omega^m \cap T)\text{succ}_{T'}(s) \neq \text{succ}_T(s) \Rightarrow \\ &(\forall a_1, \dots, a_k \in r_m)|\{t \in \text{succ}_{T'}(s) : \{a_1, \dots, a_k\} \subseteq x_{t(m)}^m\}| \geq k. \end{aligned}$$

Observe that if $T'' \leq_{k'} T' \leq_k T$ and $k \leq k'$, then $T'' \leq_k T$.

Lemma 3.5. Suppose that $T^{n+1} \leq_{k_n} T^n$ for $n \in \omega$, where $\langle k_n : n \in \omega \rangle$ is an increasing sequence. Then

$$T = \bigcap T^n \in \mathbb{P} \quad \text{and} \quad T \leq_{k_n} T^n \text{ for each } n.$$

Proof. First show by induction on the levels that T must be a perfect tree; we show that $T \in \mathbb{P}$.

Fix $n \in \omega$, we need to find \bar{m} such that:

$$\begin{aligned} &(\forall m \geq \bar{m})(\forall a_1, \dots, a_n \in r_m)(\forall s \in \omega^m \cap T) \\ &|\{t \in \text{succ}_T(s) : \{a_1, \dots, a_n\} \subseteq x_{t(m)}^m\}| \geq n. \end{aligned}$$

Choose \bar{n} such that $k_{\bar{n}} > n$ and fix \bar{m} such that

$$\begin{aligned} &(\forall m \geq \bar{m})(\forall s \in \omega^m \cap T^{\bar{n}})(\forall a_1, \dots, a_n \in r_m) \\ &|\{t \in \text{succ}_{T^{\bar{n}}}(s) : \{a_1, \dots, a_n\} \subseteq x_{t(m)}^m\}| \geq n. \end{aligned}$$

We claim that \bar{m} works; indeed fix $m \geq \bar{m}$, $s \in \omega^m \cap T$, and $a_1, \dots, a_n \in r_m$. Now $s \in T^{\bar{n}}$ but we can also find $\bar{n} \geq \bar{n}$ such that $\text{succ}_T(s) = \text{succ}_{T^{\bar{n}}}(s)$; but the result follows since $T^{\bar{n}} \leq_{k_{\bar{n}}} T^{\bar{n}}$. \square

Lemma 3.6. Assume that $T \Vdash \tau \in \omega$ and let $k \in \omega$. Then there are $T' \leq_k T$, $n \in \omega$, such that

$$(\forall s \in \omega^n \cap T')(\exists a_s \in \omega)[T'_s \Vdash \tau = a_s].$$

Proof. Let

$$S = \{t \in T : (\exists n_t \in \omega)(\exists T' \leq_k T_t)(\forall s \in T' \cap \omega^{n_t})(\exists a_s \in \omega)T'_s \Vdash \tau = a_s\}.$$

Choose \bar{m} such that

$$\begin{aligned} & (\forall m \geq \bar{m})(\forall s \in \omega^m \cap T)(\forall a_1, \dots, a_{2k} \in r_m) \\ & |\{t \in \text{succ}_T(s) : \{a_1, \dots, a_{2k}\} \subseteq x_{t(m)}^m\}| \geq 2k. \end{aligned}$$

It suffices to show that $T \cap \omega^{\bar{m}} \subseteq S$. Indeed, if we have $T(t) \leq_k T_t$ for each $t \in \omega^{\bar{m}} \cap T \subseteq S$ with associated n_t , then $T' = \bigcup_{t \in \omega^{\bar{m}} \cap T} T(t) \leq_k T$ and $n = \max\{n_t : t \in \omega^{\bar{m}} \cap T\}$ fulfills the requirements of the lemma.

Hence, fix $s \in \omega^{\bar{m}} \cap T$, and suppose for the sake of a contradiction that $s \notin S$. First of all, observe that

$$\begin{aligned} & (\forall m)(\forall t \in \omega^m \cap T)t \notin S \Rightarrow (\exists a_1, \dots, a_k \in r_m) \\ & |\{t' \in \text{succ}_T(t) \cap S : \{a_1, \dots, a_k\} \subseteq b_{t'(m)}^m\}| < k. \end{aligned}$$

We shall build $T' \leq_k T_s$ such that

$$(\forall m \geq \bar{m})[T' \cap S \cap \omega^m = \emptyset].$$

This yields the desired contradiction since we can now find $T'' \leq T'$ and $n \in \omega$ such that $T'' \Vdash \tau = n$; finally choosing $t \in T$ and $T''' \leq T''$ such that $T''' \leq_k T_t$ shows that $t \in S$!

Let $T' = \{t \in T : t \leq s \text{ or } t \geq s \text{ and } t \notin S\}$. T' is a perfect tree; we show that $T' \leq_k T_s$ and $T' \in \mathbb{P}$ under our assumption that $s \notin S$.

To verify that $T' \leq_k T_s$, fix $t \in \omega^m \cap T'$ such that $\text{succ}_{T'}(t) \neq \text{succ}_{T_s}(t)$. Since $t \notin S$, there are $a'_1, \dots, a'_k \in r_m$ such that

$$|\{t' \in \text{succ}_T(t) \cap S : \{a'_1, \dots, a'_k\} \subseteq x_{t'(m)}^m\}| < k.$$

But by assumption, since $t \in \omega^m$ and $m \geq \bar{m}$, we have that for all $a_1, \dots, a_{2k} \in r_m$,

$$|\{t' \in \text{succ}_T(t) : \{a_1, \dots, a_{2k}\} \subseteq x_{t'(m)}^m\}| \geq 2k;$$

and hence, for all $a_1, \dots, a_k \in r_m$,

$$|\{t' \in \text{succ}_T(t) \setminus S : \{a_1, \dots, a_k\} \subseteq x_{t'(m)}^m\}| \geq 2k - k = k$$

as desired.

It remains to show that $T' \in \mathbb{P}$; so we fix $n \in \omega$, and we show how to find \bar{l} such that:

$$\begin{aligned} & (\forall l \geq \bar{l})(\forall a_1, \dots, a_n \in r_l)(\forall t \in \omega^l \cap T') \\ & |\{t' \in \text{succ}_{T'}(t) : \{a_1, \dots, a_n\} \subseteq x_{t'(l)}^l\}| \geq n. \end{aligned}$$

Without loss of generality we assume that $n \geq k$. Since $T_s \in \mathbb{P}$, we have \bar{l} such that

$$\begin{aligned} & (\forall l \geq \bar{l})(\forall t \in \omega^l \cap T_s)(\forall a_1, \dots, a_{2n} \in r_l) \\ & |\{t' \in \text{succ}_{T_s}(t) : \{a_1, \dots, a_{2n}\} \subseteq x_{t'(l)}^l\}| \geq 2n. \end{aligned}$$

Now for all $t \in \omega^l \cap T'$, since again $t \notin S$, there are $a'_1, \dots, a'_k \in r_l$ such that

$$|\{t' \in \text{succ}_T(t) \cap S : \{a'_1, \dots, a'_k\} \subseteq x_{t'(l)}^l\}| < k,$$

and we then have for all $a_1, \dots, a_n \in r_l$,

$$|\{t' \in \text{succ}_T(t) \setminus S : \{a_1, \dots, a_n\} \subseteq x_{t'(l)}^l\}| \geq 2n - k \geq n$$

as desired. \square

We finally verify that \mathbb{P} adds no unbounded real. Suppose now that $T \Vdash \dot{f} \in \omega^\omega$. Build $\langle T^n : n \in \omega \rangle$ and $\langle k_n : n \in \omega \rangle$ such that $T^{n+1} \leq_{k_n} T^n$ and such that

$$(\forall s \in \omega^{k_n} \cap T^n)(\exists a_s \in \omega)[T_s^n \Vdash \dot{f}(n+1) = a_s].$$

Then $T = \bigcap_n T^n \in \mathbb{P}$ and if $g(n) = \max\{a_s : s \in \omega^{k_n} \cap T\}$, then $T \Vdash (\forall n)\dot{f}(n) \leq g(n)$ as desired. \square

4. F_σ FILTERS AND P -POINTS

We now show how to diagonalize F_σ filters while preserving P -points from the ground model.

Theorem 4.1. *It is possible to diagonalize any F_σ filter in a forcing extension while preserving all P -points from the ground model.*

The proof uses the setup of §3, and follows almost identically the construction of the partial order Q from §2 of the Blass–Shelah model [2]. Indeed, we define our partial order so closely to theirs that almost all required lemmas can be copied without change; we will only guide the reader through the very few necessary modifications.

So let us fix \mathcal{F} an F_σ filter; as in Lemma 3.2, there are a sequence $n_0 < n_1 < \dots$ and sets $a_0^k, \dots, a_{r_k-1}^k \subseteq [n_k, n_{k+1})$ for each k such that:

1. $(\forall x \in [r_k]^k) \bigcap_{i \in x} a_i^k \neq \emptyset$,
2. $(\forall X \in \mathcal{F})(\exists f \in \prod_{k \in \omega} r_k)[\bigcup\{a_{f(k)}^k : k \in \omega\} \subseteq^* X]$.

It thus suffices, in order to diagonalize \mathcal{F} , to force $Y \in [\omega]^\omega$ and $x_k \in [r_k]^k$, for $k \in Y$, such that:

$$\left(\forall f \in \prod_{k \in \omega} r_k \right)^V (\forall^\infty k \in Y)[f(k) \in x_k].$$

Notice that in §3, we obtained $Y = \omega$, but the weakening here seems necessary.

We can assume without loss of generality that $r_k > k$. Let $s_k = \binom{r_k}{k}$ and list $[r_k]^k = \{x_0^k, \dots, x_{s_k-1}^k\}$.

Define $\pi_k = \bigcup\{\prod_{i \in y} s_i : \emptyset \neq y \subseteq k\}$ and for $n < m$, let $K_{n,m}$ be the set of binary relations $t \subseteq \pi_n \times \pi_m$ such that, for each $a \in \pi_n$,

$$(a, a) \in t, \text{ and if } (a, b) \in t, \text{ then } b \upharpoonright n = a.$$

For $t \in K_{n,m}$, we define the depth $\text{Dp}(t)$ by the following induction:

$\text{Dp}(t) \geq 0$ always.

$\text{Dp}(t) \geq 1$ if, for each $a \in \pi_n$, there is $b \in \pi_m$ such that $(a, b) \in t$ and $b \neq a$.

$\text{Dp}(t) \geq d + 1$, for $d \geq 1$, if, for each $f \in \prod_{i < m} r_i$,

$$t_f := \{(a, b) \in t : (\forall i \in \text{dom}(b \setminus a))f(i) \in x_{b(i)}^i\}$$

has depth $\geq d$.

Equivalently, t has depth $\geq d + 1$ iff

$$\left(\forall f_1, \dots, f_d \in \prod_{i \in \omega} r_i \right) (\forall a \in \pi_n)(\exists b \neq a \text{ in } \pi_m) \left[(a, b) \in \bigcap_{i=1, \dots, d} t_{f_i} \right].$$

Equipped with a copy of [2], the reader will have noticed the similarities; in particular, π_n here plays the role of $\wp(n)$ there. So the partial order Q is defined similarly as the collection of all pairs $\langle w, T \rangle$ such that $T = \langle t_l : l \in \omega \rangle$ and for some increasing function $n: \omega \rightarrow \omega$ we have:

- (1) $w \in \pi_{n(0)}$,
- (2) $t_l \in K_{n(l), n(l+1)}$ for each l , and
- (3) $\text{Dp}(t_l) \rightarrow \infty$ as $l \rightarrow \infty$.

Another such condition $\langle w', T' \rangle$ is an extension of $\langle w, T \rangle$ if and only if there is an increasing function $k: \omega \rightarrow \omega$ such that, writing t_l^* for the usual composition of relations $t_{k(l)}t_{k(l+1)} \cdots t_{k(l+1)-1}$, we have:

- (1) $\langle w, w' \rangle \in t_0 t_1 \cdots t_{k(0)-1}$,
- (2) $t'_l \in K_{n(k(l)), n(k(l+1))}$ for all $l \in \omega$, and
- (3) $t'_l \subseteq t_l^*$ for all $l \in \omega$.

We show that forcing with Q does diagonalize our filter.

Lemma 4.2. *If G is Q -generic, then $g = \bigcup \{w: \langle w, T \rangle \in G\} \in \prod_{k \in Y} s_k$ for some $Y \in [\omega]^\omega$, and further,*

$$\left(\forall f \in \prod_{k \in \omega} r_k \right)^V (\forall^\infty k \in Y)[f(k) \in x_{g(k)}^k].$$

Proof. Let $f \in (\prod_{k \in \omega} r_k)^V$ and $\langle w, T \rangle \in Q$ be given, where $T = \langle t_l : l \in \omega \rangle$. Define $t'_l = (t_l)_f$ and $T' = \langle t'_l : l \in \omega \rangle$; then $\langle w, T' \rangle \leq \langle w, T \rangle$ belongs to Q and forces that

$$(\forall^\infty k \in Y)[f(k) \in x_{g(k)}^k]$$

as desired. \square

The verification that Q preserves the ground model P -points is essentially unchanged from [2], only easier.

5. CONCLUSION

We are now in position to diagonalize all filters that are included in an F_σ filter without adding an unbounded real or while preserving P -points. This will be the case for some filters of the form $\mathcal{F}(\mathcal{A})$, in particular if there are $n_0 < n_1 < \dots$ such that

$$n_{k+1} - n_k \rightarrow \infty \text{ as } k \rightarrow \infty \text{ and } (\forall A \in \mathcal{A})(\forall^\infty k)[|A \cap [n_k, n_{k+1})| \leq 1].$$

Such maximal almost disjoint families are easily constructed using Zorn's lemma by fixing a sequence $\langle n_k : k \in \omega \rangle$ as above and choosing \mathcal{A} a maximal element of the partial order, under inclusion, of all almost disjoint families satisfying the above second condition. However, using the Continuum Hypothesis, we show that not all filters of the form $\mathcal{F}(\mathcal{A})$ are included in an F_σ filter, and hence we feel that we have only grazed the problem.

Proposition 5.1 (CH). *There is a maximal almost disjoint family \mathcal{A} such that $\mathcal{F}(\mathcal{A})$ is not included in any F_σ filter.*

Proof. It suffices (and is equivalent) to build \mathcal{A} such that for all closed sets C which generate a proper filter,

$$(\exists A \in \mathcal{A})(\forall X \in C)X \cap A \text{ is infinite.}$$

Indeed, if $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{F}$, where \mathcal{F} is an F_σ filter, choose a closed set C which generates \mathcal{F} ; now our assumptions provide a contradiction.

Using CH, we list all concerned closed sets in an ω_1 sequence and we build the family \mathcal{A} by induction. At stage α , we already have defined countably many sets $\{A_n : n \in \omega\}$ from our almost disjoint family, and we built a new set A almost disjoint from the A_n 's and fulfilling our requirement for the α th closed set C_α .

If $\mathcal{F}(\{A_n : n \in \omega\})$ is not included in the filter generated by C_α , then we have nothing to do. Otherwise, we ensure that:

$$(\forall X \in C_\alpha)X \cap A \text{ is infinite.}$$

By induction, choose numbers $n_k < n_{k+1}$, $X_k \in C_\alpha$ such that

1. $X_k \subseteq^* A_k^c$,
2. $(\forall Y \in C_\alpha)Y \cap X_0 \cap \dots \cap X_k \cap [n_k, n_{k+1}) \neq \emptyset$.

This is possible by our assumptions. Finally let

$$A = \bigcup_{k \in \omega} \{X_0 \cap \dots \cap X_k \cap [n_k, n_{k+1})\}. \quad \square$$

Note however that for any almost disjoint family \mathcal{A} and $A \in \mathcal{A}$, the filter $\mathcal{F}' = \langle \mathcal{A} \setminus \{A\} \rangle$ is included in an F_σ filter since A zaps \mathcal{F}' ; but adjoining A^c can yield a filter, as above, which is not included in any F_σ filter anymore.

One might have hoped that such (maximal) almost disjoint families with the above "strengthened" maximality condition would be forced to have size continuum; however, the standard "Kunen" trick (see [3, Theorem 2.3, p. 256]) allows one to build, using CH, such a family that is indestructible by Cohen forcing. Indeed, they appear to be the likely candidates for small maximal almost disjoint families.

Added in proof. The first forcing also preserves P -points.

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