PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 114, Number 2, February 1992

ON THE JOINT SPECTRUM AND H^∞-FUNCTIONAL CALCULUS FOR PAIRS OF COMMUTING CONTRACTIONS

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(Communicated by Paul S. Muhly)

ABSTRACT. In this paper we show the existence of a pair of commuting completely nonunitary contractions (S, T) on a Hilbert space, whose joint Taylor spectrum contains the torus, such that there is a bounded analytic function h on the bidisk with $h(S, T) = 0$.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathscr{H}$ be a separable, infinite-dimensional, (complex) Hilbert space and let $\mathscr{L}(\mathscr{H})$ be the algebra of bounded linear operators on $\mathscr{H}$. Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and let $\mathbb{T} = \partial\mathbb{D}$. In [11, Chapter III], Sz.-Nagy and Foias show that if T is a contraction in $\mathscr{L}(\mathscr{H})$ that is completely nonunitary (i.e., T has no nonzero reducing subspace to which its restriction is unitary), then there is an algebra homomorphism from the algebra $H^\infty(\mathbb{D})$ of all bounded analytic functions on $\mathbb{D}$ to the algebra $\mathscr{L}(\mathscr{H})$. They then use this functional calculus to study the class C_0 of such operators T in $\mathscr{L}(\mathscr{H})$ for which there is a bounded analytic function $f: \mathbb{D} \to \mathbb{C}$ with $f(T) = 0$. This class has subsequently been extensively studied, by Sz.-Nagy and Foias and other authors. The recent book [2] by Bercovici is a comprehensive treatise on this topic.

In [4] a two variable version of the Sz.-Nagy–Foias functional calculus was developed. Therefore, one may define the analogous class $C_0^{(2)}$ of pairs (S, T) of commuting completely nonunitary contractions for which there is a bounded analytic function $h: \mathbb{D}^2 \to \mathbb{C}$ satisfying $h(S, T) = 0$. Unfortunately, this algebra $H^\infty(\mathbb{D}^2)$ of bounded analytic functions on the bidisk $\mathbb{D}^2$ is more complicated than the analogous algebra on the disk. In particular, weak* closed ideals are not necessarily principal and, also, there is no inner/outer factorization (cf. [10]). Thus, the theory of the class $C_0^{(2)}$ is much harder than in the one variable case.

In this paper we show the existence of a pair of commuting completely nonunitary contractions (S, T) in $\mathscr{L}(\mathscr{H})$, whose joint Taylor spectrum con-

Received by the editors August 10, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47A60; Secondary 47A15.
Key words and phrases. Contractions, joint spectrum, functional calculus.

tains the torus, denoted $\mathbb{T}^2$, such that $(S, T) \in C_0^{(2)}$ (Theorem 2.1). This result is analogous to [11, Corollary 5.3].

The paper is organized as follows: In §1 we set out the basic notation used throughout the paper. In §2 we construct the commuting pair (S, T) and use a result from [8] to construct the function h. In §3 we provide an alternative construction of h that does not require the result from [8]. Finally, in §4 we make some concluding remarks and pose some open problems. We will assume the reader is familiar with basic results of multivariable operator theory as presented in [7]. The following is Theorem 4.4 of [4]:

Theorem 1.1. *If S and T are commuting completely nonunitary contractions in $\mathscr{L}(\mathscr{H})$, then there is an algebra homomorphism $\Phi\colon H^\infty(\mathbb{D}^2) \to \mathscr{L}(\mathscr{H})$ with the following properties:*

(1) $\Phi(1) = I_{\mathscr{H}}$, $\Phi(w_1) = S$, $\Phi(w_2) = T$, *where w_1 and w_2 denote the coordinate functions.*
(2) $\|\Phi(h)\| \le \|h\|_\infty$, *for all $h \in H^\infty(\mathbb{D}^2)$.*
(3) Φ *is weak* continuous, (i.e., continuous when both $H^\infty(\mathbb{D}^2)$ and $\mathscr{L}(\mathscr{H})$ are given the corresponding weak* topologies).*

Remark 1.2. If the joint Taylor spectrum of (S, T) (cf. [7]), denoted by $\sigma(S, T)$, is contained in the bidisk, then the Taylor functional calculus coincides with the functional calculus of Theorem 1.1. This is a consequence of a theorem of Putinar [7, Theorem 5.20] and the fact that the image of Φ is contained in the double commutant of $\{S, T\}$. We shall write $h(S, T)$ instead of $\Phi(h)$.

We now recall the definition of the Harte spectrum (see [7]). Let (T_1, T_2) be a pair of commuting operators in $\mathscr{L}(\mathscr{H})$. We say that (T_1, T_2) is *left invertible* if there is a pair (S_1, S_2) in $\mathscr{L}(\mathscr{H})$ such that $S_1T_1 + S_2T_2 = I$. The *left Harte spectrum* is the set

$$\sigma_{lH}(T_1, T_2) = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : (T_1 - \lambda_1, T_2 - \lambda_2) \text{ is not left invertible}\}.$$

Similarly, we define *right invertible* and the *right Harte spectrum* (denoted $\sigma_{rH}(T_1, T_2)$). The *Harte spectrum* is defined by $\sigma_H(T_1, T_2) = \sigma_{lH}(T_1, T_2) \cup \sigma_{rH}(T_1, T_2)$. A theorem of Cho and Takaguchi [7, Theorem 6.8] says that $\partial\sigma(T_1, T_2) \subset \sigma_H(T_1, T_2)$. Since the Harte spectrum is contained in the Taylor spectrum, we have the following:

Lemma 1.3. *Let S and T be commuting contractions on $\mathscr{H}$. Then $\mathbb{T}^2 \subset \sigma(S, T)$ if, and only if, $\mathbb{T}^2 \subset \sigma_H(S, T)$.*

2. Construction of the pair (S, T)

Let T be a completely nonunitary contraction on $\mathscr{H}$ with $\mathbb{D} \subset \sigma(T)$. The function $h \in H^\infty(\mathbb{D}^2)$, mentioned in §1, will have the form $h(w_1, w_2) = w_1 - g(w_2)$, where $g \in H^\infty(\mathbb{D})$, maps the unit disk into itself, and has the property

$$(\dagger) \qquad \mathbb{T}^2 \subset \{(g(\lambda), \lambda) : \lambda \in \mathbb{D}\}^-.$$

The existence of such a function g follows from Theorem 1 of [8]. Since the result obtained there is much more powerful than what we need here, an alternative construction of such a g is given in §3. Given such a function g,

we use the Sz.-Nagy–Foias functional calculus to define $S = g(T)$. Clearly, S and T commute and $h(S, T) = 0$. Thus, the main result of this paper can be stated as follows:

Theorem 2.1. *Let T be a completely nonunitary contraction in $\mathscr{L}(\mathscr{H})$ such that $\mathbb{D} \subset \sigma(T)$. Then there exists a function $g \in H^\infty(\mathbb{D})$ such that $\sigma(g(T), T) \supset \mathbb{T}^2$ and clearly, $(g(T), T) \in C_0^{(2)}$.*

Proof. From property (†) and $\mathbb{D} \subset \sigma(T)$ we have that

$$\mathbb{T}^2 \subset \{(g(\lambda), \lambda) : \lambda \in \mathbb{D} \cap \sigma(T)\}^- = \{(g(\lambda), \lambda) : \lambda \in \mathbb{D}\}^-.$$

Hence, to complete the proof it is enough to show that

$$\{(g(\lambda), \lambda) : \lambda \in \mathbb{D}\}^- \subset \sigma(g(T), T).$$

This is established by the following three lemmas.

Lemma 2.2. *If $h \in H^\infty(\mathbb{D}^2)$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2$, then there exist functions $g_1, g_2 \in H^\infty(\mathbb{D}^2)$ such that for every $w = (w_1, w_2) \in \mathbb{D}^2$, we have*

$$h(w) - h(\lambda) = (w_1 - \lambda_1)g_1(w) + (w_2 - \lambda_2)g_2(w);$$

furthermore,

$$\|g_i\|_\infty \le 2\|h\|_\infty/(1 - |\lambda_i|), \qquad i = 1, 2,$$

and $\partial g_2/\partial w_1 \equiv 0$.

Proof of Lemma 2.2. The function $h(\lambda_1, w_2) - h(\lambda_1, \lambda_2)$ belongs to $H^\infty(\mathbb{D})$ and has a zero at $w_2 = \lambda_2$. Hence we can find $f_2 \in H^\infty(\mathbb{D})$ such that

$$h(\lambda_1, w_2) - h(\lambda_1, \lambda_2) = (w_2 - \lambda_2)f_2(w_2), \qquad w_2 \in \mathbb{D}.$$

Define $g_2(w_1, w_2) = f_2(w_2)$, notice that $\partial g_2/\partial w_1 \equiv 0$, and

$$\|g_2\|_\infty \le 2\|h\|_\infty/(1 - |\lambda_2|).$$

Now consider the function $h(w_1, w_2) - h(\lambda_1, w_2)$; this is an analytic function of two variables that is zero whenever $w_1 = \lambda_1$. For each fixed $w_2 \in \mathbb{D}$ we can find a function $f_{1,w_2} \in H^\infty(\mathbb{D})$ satisfying

$$h(w_1, w_2) - h(\lambda_1, w_2) = (w_1 - \lambda_1)f_{1,w_2}(w_1), \qquad w_1 \in \mathbb{D}.$$

Define $g_1(w_1, w_2) = f_{1,w_2}(w_1)$ and notice that it satisfies

$$h(w_1, w_2) - h(\lambda_1, w_2) = (w_1 - \lambda_1)g_1(w_1, w_2), \qquad (w_1, w_2) \in \mathbb{D}^2.$$

We claim g_1 is analytic as a function of each of its variables. It is clearly analytic in w_1. To see that it is analytic in w_2, first fix $w_1 \ne \lambda_1$, write the left-hand side as a power series in (w_1, w_2) about (λ_1, λ_2) (where λ_2 is an arbitrary point in $\mathbb{D}$), and observe that a factor of the form $w_1 - \lambda_1$ can be cancelled on both sides of the equation. The resulting power series is a power series expansion of g_1 (considered as a function of w_2) about λ_2. Now, use analytic continuation to show that g_1 extends analytically to points satisfying $w_1 = \lambda_1$. Also, g_1 satisfies

$$\|g_1\|_\infty \le 2\|h\|_\infty/(1 - |\lambda_1|).$$

Finally,

$$\begin{aligned} h(w) - h(\lambda) &= h(w_1, w_2) - h(\lambda_1, w_2) + h(\lambda_1, w_2) - h(\lambda_1, \lambda_2) \\ &= (w_1 - \lambda_1) g_1(w_1, w_2) + (w_2 - \lambda_2) g_2(w_1, w_2), \end{aligned}$$

and the lemma is proved. □

In [11, Theorem 5.1] it is shown that the spectrum of a contraction in the class C_0 is completely determined by its minimal function. The following lemma is a weaker version of that result for a pair of commuting contractions.

Lemma 2.3. *If S and T are commuting completely nonunitary contractions in $\mathscr{L}(\mathscr{H})$ and there is an $h \in H^\infty(\mathbb{D}^2)$ such that $h(S, T) = 0$, then*

$$\sigma_H(S, T) \cap \mathbb{D}^2 \subset \{(\lambda_1, \lambda_2) : \lambda_1 \in \sigma(S),\ \lambda_2 \in \sigma(T),\ \textit{and}\ h(\lambda_1, \lambda_2) = 0\}.$$

Proof of Lemma 2.3. If $(\lambda_1, \lambda_2) \in \sigma_H(S, T) \cap \mathbb{D}^2$, then we have $\lambda_1 \in \sigma(S)$ and $\lambda_2 \in \sigma(T)$, by the projection property of the Harte spectrum (cf. [7]). So, we just have to show that $h(\lambda_1, \lambda_2) = 0$. By Lemma 2.2, we can find $g_1, g_2 \in H^\infty(\mathbb{D}^2)$ with

$$h(w_1, w_2) - h(\lambda_1, \lambda_2) = (w_1 - \lambda_1) g_1(w_1, w_2) + (w_2 - \lambda_2) g_2(w_1, w_2);$$

hence,

$$-h(\lambda_1, \lambda_2) = (T - \lambda_1) g_1(S, T) + (S - \lambda_2) g_2(S, T).$$

If $h(\lambda_1, \lambda_2) \neq 0$, we may define $A_i = g_i(S, T)/(-h(\lambda_1, \lambda_2))$ for $i = 1, 2$, and then

$$I = A_1(T - \lambda_1) + A_2(S - \lambda_2).$$

Thus, $(\lambda_1, \lambda_2) \notin \sigma_H(S, T)$, which is a contradiction. □

The proof of the following lemma requires some knowledge of the left essential Harte spectrum (see [7]).

Lemma 2.4. *If T is a completely nonunitary contraction in $\mathscr{L}(\mathscr{H})$ and $g \in H^\infty(\mathbb{D})$, then*

$$(\ddagger) \qquad \sigma_H(g(T), T) \cap \mathbb{D}^2 = \{(g(\lambda), \lambda) : \lambda \in \sigma(T) \cap \mathbb{D}\}.$$

Proof of Lemma 2.4. Containment of the left-hand side of $(\ddagger)$ in the right-hand side follows from Lemma 2.3. We show the reverse containment. If λ is an eigenvalue of T, then the conclusion follows from the fact that $g(\lambda)$ is an eigenvalue of $g(T)$ with the same eigenvector and thus, the pair $(g(\lambda), \lambda) \in \sigma_{lH}(g(T), T)$. A similar argument is used in the case that $\bar{\lambda}$ is an eigenvalue of T^*. Hence, we can assume that λ is in the essential spectrum of T. First, assume λ is in the left essential spectrum of T. Then we can find an orthonormal sequence $\{x_n\}$ of vectors in $\mathscr{H}$ such that $\|(T - \lambda)x_n\| \to 0$. Since $T - \lambda$ is a factor of $g(T) - g(\lambda)$, we have $\|(g(T) - g(\lambda))x_n\| \to 0$. Thus, $(g(\lambda), \lambda)$ is in the left essential Harte spectrum of $(g(T), T)$. If λ is in the right essential spectrum of $g(T)$, then $\bar{\lambda}$ is in the left essential spectrum of T^* and an analogous argument shows that $(g(\lambda), \lambda)$ is in the right essential Harte spectrum of $(g(T), T)$. The lemma is proved. □

Using Theorem 1 of [8], one can actually show that there exists a function $g \in H^\infty(\mathbb{D})$ with $g(\mathbb{D}) \subset \mathbb{D}$ such than the following condition, stronger than $(\dagger)$, is satisfied:

$$\mathbb{T} \times \overline{\mathbb{D}} \subset \{(g(\lambda), \lambda) : \lambda \in \mathbb{D}\}^-.$$

Thus, the following improvement of Theorem 2.1 holds.

Theorem 2.5. *Let T be a completely nonunitary contraction in $\mathscr{L}(\mathscr{H})$ such that $\mathbb{D} \subset \sigma(T)$. Then there exists a function $g \in H^\infty(\mathbb{D})$ such that $\sigma(g(T), T) \supset \mathbb{T} \times \overline{\mathbb{D}}$ and clearly, $(g(T), T) \in C_0^{(2)}$.*

3. Alternative construction of g

The proof of Theorem 1 of [8] is highly complicated, and thus, it seems to be worthwhile to give a simpler construction of a function $g \in H^\infty(\mathbb{D})$, such that $g(\mathbb{D}) \subset \mathbb{D}$ and (†) is satisfied. Our construction of such a function g requires some knowledge of harmonic analysis and the theory of cluster sets. The information needed can be found in [9, 6].

The function g is going to be given as the composition of three functions. The first function f_1 is an analytic function on $\mathbb{D}$ that maps the disk onto an unbounded open set in the open right half plane in such a way that every radial path going to a rational point on $\mathbb{T}$ is mapped to a curve "tending to" infinity. The second function $f_2(z) = (2z-1)/(2z+1)$ is a conformal mapping of the open right half plane onto $\mathbb{D}$ such that the point at infinity corresponds to 1. The third function f_3 is a conformal mapping of $\mathbb{D}$ onto the simply connected domain $\mathbb{D}\backslash E$, (where E is an infinitely long, outward going, spiral accumulating at $\mathbb{T}$), normalized in such a way that the cluster set at 1 is the unit circle (cf. [6]). We now construct the first map.

Let $\{\lambda_n\}_{n=1}^\infty$ be an enumeration of the points on the circle of the form $e^{i\theta_n}$ with θ_n a rational between 0 and 2π. The λ_n's form a dense set in the circle. We define the function

$$u_n(z) = \operatorname{Re}\left[\frac{\lambda_n + z}{\lambda_n - z}\right].$$

For each n, the function u_n is a positive harmonic function with radial limits zero everywhere except at λ_n and $u_n(z) \to \infty$ as z tends radially to λ_n. The function u_n is just the Poisson Kernel considered as a function of z. Define

$$u = \sum_{n=1}^{\infty} 2^{-n} u_n;$$

u is also a positive harmonic function with infinite radial limit at each λ_n (the convergence of the series follows from a theorem of Harnack [9, Theorem 11.11] and the fact that $u_n(0) = 1$). Let v be a harmonic conjugate of u and define $f_1 = u + iv$. The function f_1 is an analytic function mapping $\mathbb{D}$ onto some open, unbounded subset of the open right half plane (since u is positive) and mapping every radial path going to a λ_n to a curve "tending to" infinity (i.e., $|f_1(z)| \to \infty$ as $z \to \lambda_n$ radially).

The second map has already been defined. We now say a few words about the map f_3. The image by f_3 of any curve $z(t)$, say $a < t < b$, tending to 1 has to spiral out and accumulate at $\mathbb{T}$. To see this consider the radial segment J defined by two adjacent points in the intersection of E with a radius of $\mathbb{T}$. The segment J divides the domain $\mathbb{D}\backslash E$ into two disjoint domains D_1 and D_2 one of which, say D_2, has $\mathbb{T}$ as part of its boundary. It is enough to show that if $f_3(z(a)) \in D_1$, then $f_3(z(t))$ crosses J at least once and, at most a finite, odd number of times, and stays in D_2 after the last cross. This follows from the fact that $f_3^{-1}(J)$ is a cross-cut (cf. [6]) of $\mathbb{D}$ and $z(t)$ crosses it and

has to stay in $f_3^{-1}(D_2)$ after a certain point. Now use the fact that f_3 is one to one from $\mathbb{D}$ onto $\mathbb{D}\backslash E$.

That g satisfies property (†) of §2 follows from the fact that the cluster set of g at each λ_n is $\mathbb{T}$. A radial path to λ_n is mapped by f_1 to a curve tending to infinity. This curve is mapped by f_2 to a curve tending to 1. The image of this curve by f_3 spirals out, and accumulates at $\mathbb{T}$. Hence, given any $\beta \in \mathbb{T}$ we can find a sequence $\{w_k\} \in D$ converging radially to λ_n such that $g(w_k) \to \beta$. For any $\alpha, \beta \in \mathbb{T}$, we want to find a sequence $\{w_k\} \in \mathbb{D}$ converging to α such that $g(w_k) \to \beta$. If $\alpha = \lambda_n$, for some n, we are done. Otherwise, approximate α by a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$, and for each k, find a sequence $\{w_j(k)\} \in \mathbb{D}$, such that $w_j(k) \to \lambda_{n_k}$ and $g(w_j(k)) \to \beta$ as $j \to \infty$. Now use a "diagonalization" argument to find a sequence $\{w_j\} \in \mathbb{D}$ such that $w_j \to \alpha$ and $g(w_j) \to \beta$ as $j \to \infty$. This proves (†).

4. Some open problems

There are many interesting open questions concerning the class $C_0^{(2)}$. In particular, one could ask how much of the theory developed in [11, Chapter III; 2] has an analog in the two variable case. In this section we turn our attention to some related questions concerning the functional calculus.

Let $\mathscr{C}_1(\mathscr{H})$ be the ideal of trace class operators in $\mathscr{L}(\mathscr{H})$. It is well known that the Banach space $\mathscr{C}_1(\mathscr{H})$ is the predual of $\mathscr{L}(\mathscr{H})$ and, hence, provides $\mathscr{L}(\mathscr{H})$ with a weak* topology. A subalgebra $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ is called a *dual algebra* if it contains the identity on $\mathscr{H}$ and it is weak* closed. Let T be a contraction in $\mathscr{L}(\mathscr{H})$; we denote by $\mathscr{A}_T$ the smallest dual algebra containing T. The Sz.-Nagy–Foias functional calculus is an algebra homomorphism from $H^\infty(\mathbb{D})$ into $\mathscr{A}_T$ (cf. [3]). The study of dual algebras has provided much information about the structure of contractions. One of the earliest results in this direction is the following theorem of Apostol [1, Theorem 2.2]:

Theorem 4.1. *If T is a completely nonunitary contraction on $\mathscr{H}$ with $\mathbb{T} \subset \sigma(T)$ and T has no (nontrivial) hyperinvariant subspace, then the Sz.-Nagy–Foias functional calculus is an isometry of $H^\infty(\mathbb{D})$ onto $\mathscr{A}_T$.*

The functional calculus of Theorem 1.1 provides a way of studying the dual algebra generated by a pair of commuting completely nonunitary contractions (S, T). Very little, other than Lomonosov's theorem, is known about the existence of (common, nontrivial) invariant subspaces of such pairs. In particular, the extension of Apostol's theorem to the case of two commuting contractions is an open problem:

Problem 4.2. Let (S, T) be a pair of commuting completely nonunitary contractions in $\mathscr{L}(\mathscr{H})$. If S and T have no (nontrivial) common invariant subspace, is the functional calculus of Theorem 1.2 necessarily an isometry?

In the case of the class of pairs of operators (S, T) defined in §2 above, the $H^\infty(\mathbb{D}^2)$-functional calculus is not an isometry (it has a kernel), but a beautiful theorem of Brown, Chevreau, and Pearcy [5, Corollary 1.2] implies that T has a (nontrivial) invariant subspace. Since $S = g(T)$ is a weak* limit of polynomials in T, this subspace is also invariant under S, and is thus, a (common, nontrivial) invariant subspace for the pair (S, T).

Let us now consider a commuting pair of operators (S, T) that satisfies the relations $S^2 = T^2$ and $\sigma(T) \supset \mathbb{T}$. Then $(S+T)(S-T) = 0$, and either $T = \pm S$, or $\ker(S-T)$ and $\ker(S+T)$ are proper subspaces of $\mathscr{H}$, at least one of which is nonzero. Without loss of generality assume $\mathscr{M} = \ker(S-T) \neq (0)$. The subspace $\mathscr{M}$ is, clearly, a (common, nontrivial) hyperinvariant subspace for the pair (S, T) (i.e., a subspace invariant for any operator in the commutant of $\{S, T\}$). Thus, the relation $T^2 - S^2 = 0$ together with $\sigma(T) \supset \mathbb{T}$ implies the existence of a (common, nontrivial) invariant subspace for the pair (S, T), but this argument does not work if the relation $S^2 = T^2$ is replaced by other relations (e.g. $T^3 - S^2 = 0$). This raises the following:

Problem 4.3. Let (S, T) be a pair of commuting completely nonunitary contractions whose Taylor spectrum contains the torus. If $(S, T) \in C_0^{(2)}$ do S and T have a (nontrivial) common invariant subspace?

The results in this note constitute part of the author's Ph.D. thesis at the University of Michigan, written under the direction of Professor Carl Pearcy.

References

1. C. Apostol, *Ultraweakly closed operator algebras*, J. Operator Theory **2** (1979), 49–61.
2. H. Bercovici, *Operator theory and arithmetic in H^∞*, Math. Surveys Monographs, Amer. Math. Soc., Providence, RI, 1988.
3. H. Bercovici, C. Foias, and C. Pearcy, *Dual algebras with applications to invariant subspaces and dilation theory*, CBMS Regional Conf. Series in Math., vol. 56, Amer. Math. Soc., Providence, RI, 1985.
4. E. Briem, A. M. Davie, and B. K. Øksendal, *Functional calculus for commuting contractions*, J. London Math. Soc. (2) **7** (1973), 709–718.
5. S. Brown, B. Chevreau, and C. Pearcy, *On the structure of contraction operators* II, J. Funct. Anal. **76** (1988), 30–55.
6. E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*, Cambridge Univ. Press, London, 1966.
7. R. E. Curto, *Applications of several complex variables to multiparameter spectral theory*, Surveys of Recent Results in Operator Theory, vol. II, Longman, London, 1988, pp. 25–90.
8. A. J. Lohwater and G. Piranian, *Bounded analytic functions with large cluster sets*, Ann. Acad. Sci. Fenn. Ser. A I **499** (1971), 3–6.
9. W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill, New York, 1987.
10. ——, *Function theory on polydisks*, Benjamin, New York, 1969.
11. B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, 1970.

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