ON GROUPS WITH A CENTRAL AUTOMORPHISM OF INFINITE ORDER

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Abstract. It is shown that a group $G$, whose center has finite exponent, has a central automorphism of infinite order if and only if $G$ has an infinite abelian direct factor. It is also shown that the group of central automorphisms of a nilpotent $p$-group of infinite exponent contains an uncountable torsionfree abelian subgroup.

1. Introduction

In their paper [5] Menegazzo and Stonehewer show that, apart from a few obvious exceptions, a nilpotent $p$-group always has an outer automorphism of order $p$. They also observe that it is often easier, in the case of nilpotent $p$-groups, to construct automorphisms of finite order and, therefore, pose the question of which nilpotent $p$-groups have such an automorphism. A partial answer to this question is stated in [5] and an example of a nilpotent $p$-group that has no automorphism of infinite order is given.

The purpose of the current paper is to characterize those nilpotent $p$-groups $G$ that have a central automorphism of infinite order. (Here an automorphism of $G$ is called central if it acts trivially on the group modulo its center. We denote the group of central automorphisms of $G$ by $\text{Aut}_c G$.) There are two cases to consider. We show in §2 that if $G$ is a nilpotent $p$-group of infinite exponent, then $\text{Aut}_c G$ contains an uncountable torsionfree abelian subgroup. The arguments used here follow those of Buckley and Wiegold [1, Theorems 2.2, 2.6], for the most part. However, some additional results are needed, because the automorphisms constructed in [1] do not always have infinite order. In §3 we obtain our main result concerning central automorphisms of nilpotent $p$-groups. We show that if $G$ is such a group of finite exponent, then $G$ has a central automorphism of infinite order if and only if $G$ has an infinite abelian direct factor. However, our result is a corollary to the following much more general theorem, which we prove in §3.

Theorem 3.1. Suppose $G$ is a group and $Z(G)$ has finite exponent. Then the following are equivalent.

(i) $G$ has a central automorphism of infinite order.
(ii) \( G \) has an infinite abelian direct factor.
(iii) \( \text{Aut}_c G \) contains an uncountable torsionfree abelian subgroup.

Our notation is standard for the most part. For any group \( G \), the commutator subgroup is \( G' \), the center is \( Z(G) \), which we abbreviate to \( Z \), and we write \( G_{ab} \) for \( G/G' \). We use additive notation for abelian groups and, in particular, for subgroups of \( Z \). For a prime \( p \) and natural number \( n \), \( Z[p^n] \) denotes the subgroup of elements of \( Z \) of order dividing \( p^n \), and \( p^nZ \) denotes the subgroup of \( Z \) consisting of \( \{p^n z | z \in Z \} \). We also recall that if \( N \leq Z \) then \( \text{Hom}(G/N, N) \) can be identified with the subgroup of elements of \( \text{Aut}_c G \) that act trivially on both \( N \) and \( G/N \).

2. Nilpotent \( p \)-groups of infinite exponent

Throughout this section we let \( G \) be a nilpotent \( p \)-group of infinite exponent. We refer the reader to [1] and [8] for standard facts concerning basic subgroups of nilpotent \( p \)-groups; recall that a subgroup \( B \) of the group \( G \) is called basic if \( G' \leq B \) and \( B/G' \) is a basic subgroup of the abelian group \( G_{ab} \). We require two preliminary results before proving the main result of this section.

**Lemma 2.1.** Suppose \( B \) is a basic subgroup of \( G \) and \( N \leq Z(G) \). Suppose \( G = BN \) and \( B \cap N \) has finite exponent. Then \( \text{Aut}_c G \) contains an uncountable torsionfree abelian subgroup.

**Proof.** Let \( Q = \frac{A}{B \cap N} \), so \( Q \) is a periodic radicable abelian group. Because \( B \cap N \) has finite exponent, it follows that \( \text{Ext}(Q, B \cap N) \) has finite exponent \( m \), say and clearly \( m \) is a power of \( p \). If \( \pi \) is a \( p \)-adic integer of the form \( 1 + pm + s_2p^2 + s_3p^3 + \ldots \), where each \( s_i \) is either 0 or \( m \) and infinitely many of them are nonzero, then multiplication by \( \pi \) induces an automorphism of \( Q \) and by [4, Lemma 52.1], \( \pi^* \) is multiplication by \( \pi \) on \( \text{Ext}(Q, B \cap N) \). Thus if \( \Delta \in \text{Ext}(Q, B \cap N) \),

\[
\pi^* \Delta = \Delta \quad \text{because } m \Delta = 0.
\]

On the other hand, because \( Q \) is divisible and \( B \cap N \) has finite exponent, the theory of automorphisms of group extensions (see [7, p. 70]) shows that

\[
\text{C}_{\text{Aut} Q}(\Delta) \leq \text{Aut} N.
\]

Hence \( \text{Aut} N \) contains an uncountable, torsionfree abelian subgroup \( A \), fixing \( B \cap N \). Finally if we define, for each \( \alpha \in A \),

\[
(bn)\bar{\alpha} = b(n\alpha) \quad \text{for } b \in B, \quad n \in N,
\]

then \( \bar{\alpha} \in \text{Aut} G \) and it follows that \( \text{Aut} G \) contains a subgroup isomorphic to \( A \). \( \square \)

The Cartesian product of a family \( \{G_i : i \in I\} \) is denoted by \( \bigtimes_{i \in I} G_i \) and the direct sum by \( \bigoplus_{i \in I} G_i \). For a natural number \( n \), we let \( C_n \) be the cyclic group of order \( n \), and for the prime \( p \), we let \( C_{p^n} \) be the quasicyclic \( p \)-group.

**Lemma 2.2.** Suppose \( I \) is an index set and \( B = \bigoplus_{i \in I} C_{p^{n(i)}} \). Suppose that \( C \) is a subgroup of \( B \) of infinite exponent. Then there is a direct sum decomposition \( B = X \oplus Y \) such that both \( X \cap C \) and \( Y \cap C \) have infinite exponent.

**Proof.** We may assume \( D = \bigoplus_{i=1}^{\infty} C_{p^{n(i)}} \), where \( n(i) \leq n(j) \) if \( i \leq j \), is a subgroup of \( B \) such that \( E = C \cap D \) has infinite exponent. Let \( (c_1, \ldots, c_1, k_1, \ldots) \),
0, ...), (c_{21}, ..., c_{2, k_2}, 0, ...), ... be a sequence of elements of \( E \) with orders \( p^{m_1}, p^{m_2}, ... \) where

\[ p^{m_{i+1}} > p^{n(k_i)+m_i} \quad i = 1, 2, ... . \]

Then \( k_i < k_{i+1} \) for all \( i \). Consider

\[ p^{n(k_i)}(c_{i+1, 1}, ... , c_{i, k_i}, ... , c_{i+1, k_{i+1}}, 0, ...). \]

This is an element of \( E \) of order \( p^{m_{i+1} - n(k_i)} > p^{m_i} \). Set \( I_1 = \{1, ..., k_1\} \) and \( I_{j+1} = \{k_j + 1, ..., k_{j+1}\} \) for \( j \geq 1 \). We may find sets \( J, K \subseteq I \) such that \( J \cup K = I, J \cap K = \emptyset, \bigcup_{j=1}^{\infty} I_{2j} \subseteq J \), and \( \bigcup_{j=1}^{\infty} I_{2j-1} \subseteq K \). Then set \( X = \bigoplus_{i \in J} C_{p^{n(i)}} \) and \( Y = \bigoplus_{i \in K} C_{p^{n(i)}} \). The subgroups \( X \) and \( Y \) have the desired properties. \( \square \)

**Theorem 2.3.** If \( G \) is a nilpotent \( p \)-group of infinite exponent then \( \text{Aut}_eG \) contains an uncountable torsionfree abelian subgroup.

**Proof.** There are a number of cases to consider.

Case 1. Suppose \( G \) is nonreduced, and let \( B \) be a basic subgroup of \( G \). Let \( N \) be a central subgroup of \( G \) such that \( N \cong C_p^\infty \). If \( G/BN \) is nontrivial, then \( \text{Hom}(G/BN, N) \) is an uncountable torsionfree abelian subgroup of \( \text{Aut}_eG \) as required. If \( G = BN \) then either \( N \leq B \) or \( B \cap N \) is finite. In the former case, \( G_{ab} = \bigoplus_{i \in I} C_{p^{n(i)}} \), for some index set \( I \) and \( G_{ab} \) has infinite exponent, since \( G \) does. Since \( G_{ab} \) is reduced, \( N \leq G' \) and hence \( \text{Hom}(G_{ab}, N) \leq \text{Aut}_eG \). Thus \( C_{p^{n(i)}} \) is a subgroup of \( \text{Aut}_eG \) and the result now follows from [1, Lemma 2.5]. If \( B \cap N \) is finite then Lemma 2.1 applies, again giving the result.

Case 2. Suppose \( G \) is reduced. According to [8, XVI] every basic subgroup \( B \) is infinite. If \( B \) has finite exponent then [8, XV] shows that \( G = BZ \). Hence by Lemma 2.1, \( \text{Aut}_eG \) satisfies the desired conclusion. If \( B \) has infinite exponent but \( G' \cap Z \) has finite exponent, then \( G/Z \) also has finite exponent (the correct version of [8, IX]). Hence \( B \cap Z \) has infinite exponent. Then \( (B \cap Z)G'/G' \) is a subgroup of infinite exponent in the group \( G/G' \), a direct sum of cyclic groups. By Lemma 2.2, we can find \( X/G', Y/G' \) so that \( B/G' = X/G' \oplus Y/G' \) and \( X/G' \cap (B \cap Z)G'/G', Y/G' \cap (B \cap Z)G'/G' \) both have infinite exponent.

Hence \( X \cap Z \) and \( Y \cap Z \) both have infinite exponent. However, \( B/X \) is a basic subgroup of \( G/X \) and the epimorphism of Széle [4, 36.1] yields

\[ \text{Hom}(Y/G', X \cap Z) \leq \text{Hom}(G/X \cap Z, X \cap Z) \leq \text{Aut}_eG \]

and \( \text{Aut}_eG \) contains an uncountable torsionfree abelian subgroup in this case.

Finally, if both \( B \) and \( G' \cap Z \) have infinite exponent then the proof of [1, Theorem 2.6] gives the result in this case. This completes the proof. \( \square \)

**3. THE FINITE EXPONENT CASE**

We first give a proof of Theorem 3.1.

Let \( \pi \) denote the set of primes dividing the exponent of \( Z(G) \).

(i) **implies** (ii). Let \( \alpha \) be a central automorphism of \( G \) that has infinite order and note that \( \alpha \) induces an automorphism of infinite order on \( Z = Z(G) \). (We say that \( \alpha \) acts infinitely on \( Z \).) Since \( \pi \) is finite and, for each prime \( p \in \pi \), the Sylow \( p \)-subgroups of \( Z \) are characteristic in \( G \), there is some prime \( p \in \pi \).
such that $\alpha$ acts infinitely on the Sylow $p$-subgroup of $Z$, which we denote by $K$. Let $L$ be the $p'$-part of $Z$ so that $Z = K \oplus L$.

Let $k \in \mathbb{N}$ be minimal such that some nontrivial power of $\alpha$ acts trivially on $p^k K$. Replacing $\alpha$ with a nontrivial power of itself if necessary we may assume that $\alpha$ acts trivially on $p^k K$. It then follows that $\alpha$ acts infinitely on $p^{k-1} K / p^k K$, otherwise some nontrivial power of $\alpha$ acts trivially on both $p^{k-1} K / p^k K$ and $p^k K$. However, this implies some nontrivial power of $\alpha$ acts trivially on $p^{k-1} K$, contrary to the minimality of $k$.

We set $M = p^{k-1} K (\alpha - 1) = \{ z(\alpha - 1) | z \in p^{k-1} K \}$ and note that $M$ is an $\alpha$-invariant subgroup of $p^{k-1} K$. To complete the proof we now establish a series of claims.

(a) $M$ has exponent $p$.

For if $m \in M$, then $m = za - z$ for some $z \in p^{k-1} K$. Then $pz \in p^k K$ so $pz = p(za) = p(za) = p(z) = p(za) = p(za) = p(za) = p(za) = p(za) = p(za)$.

(b) $\alpha$ acts infinitely on $M$.

For if $\alpha$ acts finitely on $M$ then some nontrivial power of $\alpha$ acts trivially on $M$. Clearly $\alpha$ acts trivially on $p^{k-1} K / M$ so some power of $\alpha$ acts trivially on $p^{k-1} K$, which contradicts the choice of $k$.

(c) $\alpha$ acts infinitely on $P = MG'(p^k K) / G'(p^k K)$. Otherwise $\alpha$ acts finitely on $P$, so some nontrivial power of $\alpha$ acts trivially on $P$. Since $\alpha$ is central, it acts trivially on $G'$. Hence some nontrivial power of $\alpha$ acts trivially on $MG'(p^k K)$, contrary to (b). Hence (c) follows.

We let $C = C_p(\alpha)$. It follows from (a) that $P = C \oplus D$ for some subgroup $D$ (which is not necessarily $\alpha$-invariant.) Furthermore $D$ is infinite by (c). Let $I$ be an index set and, for $i \in I$, choose $r_i \in M$ so that $\{ r_i G'(p^k K) | i \in I \}$ is a basis of $D$. Define $N = \langle r_i | i \in I \rangle$ and note that $N$ is an infinite elementary abelian subgroup of $M \leq p^{k-1} K$. It is clear that:

(d) $n^\alpha \not\equiv n \mod G'(p^k K)$ for all nontrivial $n \in N$.

It follows from [4, p. 119, Example 5] that $K$ (and hence $Z$) has a direct summand $A$ such that $A[p] = N$. Set $Z = A \oplus F$. Since $A$ is a direct summand of $Z$, each $0 \neq n \in N$ has the same $p$-height in $A$ as in $Z$. Furthermore $N \leq M \leq p^{k-1} K$. Hence:

(e) The $p$-height of $0 \neq n \in N$ in $A$ is at least $k - 1$. (In fact one can easily see that it is exactly $k - 1$.)

(f) $AG'/G'$ is a direct summand of $G_{ab}$.

It suffices by [4, Corollary 27.5] to show that $AG'/G'$ is pure in $G_{ab}$, and to do this it is enough to show that every element of order $p$ in $AG'/G'$ has the same $p$-height in $G_{ab}$ as in $AG'/G'$ (see [4, p. 114, (h)]) and also note that the Sylow $p$-subgroup of $G_{ab}$ is pure in $G_{ab}$).

Now $A \cap G' = 1$ since $\alpha$ fixes no nontrivial element of $A[p] = N$, by (d), whereas the central automorphism $\alpha$ acts trivially on $G'$. It follows that $(AG'/G')[p] = NG'/G'$.

Suppose for a contradiction that $n + G' \in NG'/G'$ has larger $p$-height in $G_{ab}$ than in $AG'/G'$. Then by (e), $n + G' = p^l g + G'$ for some $g \in G$ and some $l \geq k$. Since $\alpha$ is central, $g \alpha = g + z$ for some $z \in Z$ so that

(1) \[ n \alpha + G' = (p^l g) \alpha + G' = p^l g + p^l z + G' = n + p^l z + G'. \]

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Also \( pn = 0 \) so \( p^{l+1}g \in G' \) and since \( \alpha \) is central \( p^{l+1}g = (p^{l+1}g)\alpha = p^{l+1}g + p^{l+1}z \). Hence \( z \in K \), the Sylow \( p \)-subgroup of \( Z \) so (1) shows
\[
n\alpha \equiv n \mod G'p^kK,
\]
contrary to (d). This proves (f).

The result now follows immediately. For there is a subgroup \( H \) of \( G \) such that
\[
G_{ab} = AG'/G' \oplus H/G'.
\]
Hence \( G = AH \). But \( A \cap H \leq A \cap G' = 1 \) and \( A \leq Z(G) \) so it follows that \( G = A \times H \).

(ii) implies (iii). Clearly \( Cr^\infty_{m=1}C_{p^m} \leq \text{Aut}_cG \). Hence by [1, Lemma 2.5] the result follows.

(iii) implies (i). This is clear.

In particular, we have

**Corollary 3.2.** Suppose \( G \) is a nilpotent \( p \)-group of finite exponent. Then \( G \) has a central automorphism of infinite order if and only if \( G \) has an infinite abelian direct factor.

This result is analogous to [3, Lemma 2] although the proof is rather different. The reader is also referred to [2, 6] where further results have been obtained on central automorphisms of infinite groups.

**An example.** We note that Theorem 3.1 fails if \( Z(G) \) is allowed to have infinite exponent. For each prime \( p \), let \( G_p \) be a nilpotent \( p \)-group as in [5, 3.2(ii)] such that \( \text{Aut} G_p = \text{Aut}_cG_p \) is an elementary abelian \( p \)-group. Set \( G = \bigoplus G_p \), the direct sum being taken over all primes \( p \). Then \( \text{Aut} G = \text{Cr}(\text{Aut} G_p) \) and \( \text{Aut}_cG \) clearly contains elements of infinite order. However \( G \) has no infinite abelian direct factor.

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**References**


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