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UNIFORM AND SOBOLEV EXTENSION DOMAINS

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ABSTRACT. We prove that if a domain $D \subset \mathbf{R}^n$ is quasiconformally equivalent to a uniform domain, then D is an extension domain for the Sobolev class W_n^1 if and only if D is locally uniform. We provide examples which suggest that this result is best possible. We exhibit a list of equivalent conditions for domains quasiconformally equivalent to uniform domains, one of which characterizes the quasiconformal homeomorphisms between uniform and locally uniform domains.

1. INTRODUCTION

This article concerns three classes of domains D in Euclidean n-space $\mathbf{R}^n$. We call D a W_p^1-*extension* domain if there exists a bounded linear extension operator from $W_p^1(D)$ to $W_p^1(\mathbf{R}^n)$; here $p \geq 1$ and $W_p^1(D)$ denotes the Sobolev space of measurable functions $u : D \to \mathbf{R} \cup \{-\infty, \infty\}$ satisfying

$$||u|| = \left(\int_D |u|^p\right)^{1/p} + \left(\int_D |\nabla u|^p\right)^{1/p} < \infty ,$$

where ∇u represents the distributional gradient of u.

O. Martio and J. Sarvas [MS] introduced the notion of a *uniform* domain; see [HK] and the references mentioned there for properties of this important class of domains. We call D *c-uniform* provided each pair of points $x, y \in D$ can be joined by a rectifiable arc $\gamma \subset D$ satisfying

$$(1) \qquad \begin{cases} l(\gamma) \leq c|x-y| & \text{and} \\ \min\{l(\gamma'), l(\gamma'')\} \leq c\,\mathrm{dist}(z, \partial D) & \text{for all } z \in \gamma. \end{cases}$$

Here $l(\gamma)$ is the Euclidean arclength of γ and γ', γ'' are the components of $\gamma \setminus \{z\}$. Condition (1) asserts that the length of γ is comparable to the distance between its endpoints and that away from its endpoints γ stays away from the boundary ∂D of D. In particular, (1) implies that points can be joined in D with a curvilinear double cone which is not too crooked nor too thin.

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Next, D is (c, r)-*locally uniform* if points $x, y \in D$ with $|x-y| \leq r$ can be joined by a rectifiable arc $\gamma \subset D$ satisfying (1). Every uniform domain is locally uniform; the converse holds for domains D with $\operatorname{diam}(\partial D) < \infty$ [HK, 3.4]. P. W. Jones introduced the class of locally uniform domains and established the following fundamental result relating them and W_p^1-extension domains [J1, Theorems 1, 3].

Fact. *A (c, r)-locally uniform domain $D \subset \mathbf{R}^n$ is a W_p^1-extension domain for all $p \geq 1$ and the norm of the extension operator is bounded by a constant which depends only on c, r, p, n, and* $\operatorname{diam}(D)$. *Conversely, a finitely connected W_2^1-extension domain $D \subset \mathbf{R}^2$ is locally uniform.*

This yields no information about infinitely connected W_2^1-extension domains in $\mathbf{R}^2$ and nothing for W_n^1-extension domains in $\mathbf{R}^n$. Recently we verified that certain infinitely connected plane domains are W_2^1-extension domains if and only if they are locally uniform [HK, 7.11]. Previously Koskela established numerous properties of W_p^1-extension domains [K, §6].

As every finitely connected plane domain is conformally equivalent to a domain whose boundary components are circles or points, the following generalizes Jones' theorem. Koskela proved this for the case where $\operatorname{diam}(\partial D) < \infty$ [K, 6.3(ii)], but neither his technique nor Jones' can be used to establish our result.

Theorem. *Suppose $D \subset \mathbf{R}^n$ is quasiconformally equivalent to a uniform domain. Then D is a W_n^1-extension domain if and only if D is locally uniform. Moreover, all constants depend only on the given data.*

We explain our proof in §3 and exhibit examples in §4 which illustrate why this theorem is essentially best possible. §2 contains preliminary information, some of which may be of independent interest. In §5 we present a list of equivalent conditions for certain domains, one of which characterizes the quasiconformal homeomorphisms of locally uniform domains onto uniform domains.

Our notation and terminology conform with that of [HK]. In particular, $B(x, r)$ and $S(x, r) = \partial B(x, r)$ are the open ball and sphere of radius r centered at x. We write $c = c(a, \dots)$ to indicate that c depends only on the parameters $a, \dots$.

2. Preliminaries

Here we collect some facts used in our proof. First we state the following obvious sufficient condition for local uniformity; see the proofs of [HK, 3.7, 7.1].

2.0. **Fact.** *Suppose there exist constants c, r such that for each $z \in \partial D \setminus \{\infty\}$ there is a c-uniform domain G, with $D \cap B(z, r) \subset G \subset D$. Then D is (b, t)-locally uniform where b, t depend only on c, r.*

Next we list certain properties of uniform domains: a geometric localization result due to P. W. Jones [J2] and a consequence of work of P. Tukia and J. Väisälä [TV, 2.8, 2.9, 2.15], [V, 3.2, 4.11].

2.1. **Fact.** *Suppose $D \subset \mathbf{R}^n$ is c-uniform.*

(a) *There exists a constant $d = d(c, n)$ such that for each $z \in \partial D \setminus \{\infty\}$*

and for all $t > 0$, *there is a* d*-uniform domain* G *with* $D \cap B(z, t/d) \subset G \subset D \cap B(z, t)$.

(b) *If* $g : D \to D'$ *is a homeomorphism and there exists a constant* h *such that* $|g(u) - g(v)| \leq h|g(w) - g(v)|$ *for all* $u, v, w \in D$ *with* $|u - v| \leq |w - v|$, *then* D' *is* b*-uniform where* $b = b(c, h, n)$.

The only property of W_n^1-extension domains we require for our proof is the following geometric condition [K, 5.7, 5.8, 5.10].

2.2. **Fact.** *Suppose* $D \subset \mathbf{R}^n$ *is a* W_n^1*-extension domain. Then there exist constants* a *and* ρ, *depending only on* n *and the norm of the extension operator, such that*

$$\text{(2)} \qquad \begin{cases} \text{for all } x \in \mathbf{R}^n \text{ and all } 0 < r \leq \rho, \text{ points in } D \cap \overline{B}(x, r) \\ (\text{respectively, } D \setminus B(x, r)) \text{ can be joined by a continuum in} \\ D \cap \overline{B}(x, ar) \ (\text{respectively, } D \setminus B(x, r/a)). \end{cases}$$

In a forthcoming paper we generalize a result of F. W. Gehring and O. Martio [GM, 3.1] which has the following corollary. Here $C(f, x)$ denotes the *cluster set* of f at x.

2.3. **Fact.** *Let* $f : D \to D'$ *be* K*-quasiconformal. Suppose* $D \subset \mathbf{R}^n$ *satisfies* (2) *and* $D' \subset \mathbf{R}^n$ *is* c*-uniform. Then*:

(a) f^{-1} *has a continuous extension to* $\overline{D}'$ *and* f *has a continuous one-to-one extension to* $\overline{D} \setminus \{\infty\}$.

(b) *Assume that either* D *and* D' *are bounded or that* $\infty \in \partial D \cap C(f, \infty)$. *Then for each* $k > 0$ *there is an* $h = h(k, a, c, K, n) > 1$ *such that for all* $u, v, w \in D$ *we have* $|f(u) - f(v)| \geq k|f(w) - f(v)|$ *whenever* $\operatorname{diam}\{u, v, w\} \leq \rho$ *and* $|u - v| \geq h|w - v|$.

3. Proof of theorem

It suffices to verify that a W_n^1-extension domain quasiconformally equivalent to a uniform domain is in fact locally uniform.

Fix $D \subset \mathbf{R}^n$ and suppose there exists a linear extension operator from $W_n^1(D)$ to $W_n^1(\mathbf{R}^n)$ with norm N. Assume D' is c-uniform and $f : D \to D'$ is K-quasiconformal. We demonstrate that D is (b, r)-locally uniform where $b = b(N, K, c, n)$, $r = r(N, K, c, n)$. By 2.0 it suffices to exhibit such constants b, r which enjoy the property that for each $z \in \partial D \setminus \{\infty\}$ there exists a b-uniform domain G satisfying

$$D \cap B(z, r) \subset G \subset D. \tag{3}$$

Using 2.2 we get constants $a = a(N, n)$ and $\rho = \rho(N, n)$ so that (2) holds. According to [K, 6.3] and the Möbius invariance of uniform domains, we can assume that $\infty \in \partial D \cap \partial D'$ and $f^{-1}(\infty) = \infty$. We are now in position to take advantage of 2.3.

Let $r = \rho/3h$ where $h = h(N, K, c, n) \geq 1$ is obtained via 2.3(b) using $k = d$, and $d = d(c, n)$ comes from 2.1(a) applied to D'. Fix $z \in \partial D \setminus \{\infty\}$ and let $z' = f(z)$. Employing 2.3(a) we see that $z' \neq \infty$ and that

$$t = \operatorname{dist}(z', f(\overline{D} \cap S(z, \rho/2))) > 0.$$

Then 2.1(a) ensures the existence of a d-uniform domain G' satisfying

$$D' \cap B(z', t/d) \subset G' \subset D' \cap B(z', t).$$

Next, from 2.3(b) and 2.1(b) we conclude that $G = f^{-1}(G')$ is b-uniform with $b = b(N, K, c, n)$. It remains to verify (3).

Fix $y' = f(y) \in S(z', t) \cap f(\overline{D} \cap S(z, \rho/2))$. Let $x \in D \cap B(z, r)$ and set $x' = f(x)$. Then

$$|y - z| = \rho/2 > hr \geq h\,|x - z|,$$

whence by 2.3(b)

$$t = |y' - z'| > d\,|x' - z'|;$$

thus $x' \in D' \cap B(z', t/d) \subset G'$, so $x \in G$ and (3) holds.

4. Examples

We begin by pointing out that there are simply connected plane domains which are W_p^1-extension domains for each $p > 2$ or for each $1 \leq p < 2$, yet are not locally uniform; [M, 1.5.2], [K, 2.5, 6.8]. Furthermore, one cannot replace the quasiconformal equivalence with topological equivalence. We construct explicit examples of topological balls in $\mathbf{R}^n$ ($n \geq 3$) which are W_p^1-extension domains for all $p \geq 1$ but fail to be locally uniform; in fact, according to Jones [J1, p. 75] there are even such Jordan domains. Next, we give examples which indicate that quasiconformal equivalence to a uniform domain is necessary; one cannot weaken this, e.g., to quasiconformal equivalence to a locally uniform domain. Thus, the three hypotheses in our theorem are essential: one must assume $p = n$ and some kind of smoothness criterion is necessary, but quasiconformal equivalence to anything weaker than a uniform domain will not suffice.

4.1. **Example.** For $n \geq 3$ there exist domains in $\mathbf{R}^n$, which are homeomorphic to a ball and are W_p^1-extension domains for all $p \geq 1$, but are not locally uniform.

For notational ease we assume $n = 3$; the modifications necessary for general $n \geq 3$ are clear. Let $D = Q \setminus \bigcup I_j$ where $Q = (0, 2) \times (-1, 1) \times (0, 2)$ and $I_j = \{(1/j, 0, t) : 0 \leq t \leq 1\}$ ($j = 1, 2, \ldots$). Then D is homeomorphic to a ball. Next, as $\bigcup I_j$ has zero 2-measure, each $u \in W_p^1(D)$ can be considered as an element of $W_p^1(Q)$ (simply take an ACL representative for u and extend it to be zero on $\bigcup I_j$); since Q is an extension domain, so is D. Finally, D is not locally uniform because for any $m > 0$ we can find points $x = (\varepsilon, \varepsilon, 1/2)$, $y = (\varepsilon, -\varepsilon, 1/2)$ in D with $|x - y| \leq 1/m$ such that for any arc $\gamma \subset D$ joining x, y, either $l(\gamma) > m|x - y|$ or $l(\alpha) > m\,\mathrm{dist}(z, \partial D)$ for some $z \in \gamma$, and both components α of $\gamma \setminus \{z\}$; see 4.3 and 4.4 for similar details.

4.2. **Proposition.** *There exist domains D quasiconformally equivalent to locally uniform domains which are W_p^1-extension domains for all $p \geq 1$ but fail to be locally uniform.*

We prove 4.2 by presenting two examples of a domain D possessing the following properties.

(a) $D = \phi(\mathbf{R}^n \setminus Z)$ where ϕ is a quasiconformal self-homeomorphism of

$\overline{\mathbf{R}}^n$, and Z is the set of all points $(x_1, \ldots, x_{n-1}, 0) \in \mathbf{R}^n$ with each x_j an integer $(j = 1, \ldots, n-1)$.

(b) D satisfies (2) with $a = 1$, $\rho = \infty$.

(c) D is a W_p^1-extension domain for all $p \geq 1$.

(d) D is not locally uniform.

Note that $\mathbf{R}^n \setminus Z$ is locally uniform. As the complement of a domain D satisfying (a) has $(n-1)$-measure zero, (c) follows just as in 4.1; (b) is obvious, and we indicate why (d) is true. Obviously it is not essential that Z be such a simple set; in fact, according to [HK, 7.4] we could, e.g., replace Z by a union of closed balls each of radius $1/3$ with centers at the integer lattice points.

Our first example enjoys the property that ϕ is a Möbius transformation. Our second example is of interest because ϕ fixes the point at infinity, so ϕ is quasisymmetric in the sense of [TV]. For notational clarity we take $n = 2$ and identify $\mathbf{R}^2$ with $\mathbf{C}$.

4.3. **Example.** Let ϕ be inversion in the unit circle. Then $D = \phi(\mathbf{C} \setminus \mathbf{Z}) = \mathbf{C} \setminus \{0, \pm 1, \pm 1/2, \pm 1/3, \ldots\}$ satisfies (a)–(d) above.

To see that D is not locally uniform, let m be a positive integer and suppose that $\gamma \subset D$ is an arc joining the points $z = i/m^2$ and $w = \overline{z}$ with $l(\gamma) \leq m|z - w|$. Fix $x \in \gamma \cap \mathbf{R}$. Then $2|x| \leq l(\gamma) \leq 2/m$, so $x \in (-1/m, 1/m)$. Choose a positive integer k so that $1/(k+1) < |x| < 1/k$. Then $k \geq m$ and $\operatorname{dist}(x, \partial D) \leq 1/k(k+1)$, whence

$$l(\alpha) \geq |x| > 1/(k+1) \geq k \operatorname{dist}(x, \partial D) \geq m \operatorname{dist}(x, \partial D)$$

for either component α of $\gamma \setminus \{x\}$.

4.4. **Example.** Let $\phi(z) = z|z|^{-1/2}$. Then $D = \phi(\mathbf{C} \setminus \mathbf{Z})$ satisfies (a)–(d) above.

Again, it suffices to show that D fails to be locally uniform. To this end, let m be any positive integer and set $z = 2m + i$, $w = \overline{z}$. Suppose that γ is any arc joining z, w in D with $l(\gamma) \leq m|z - w| = 2m$. Fix $x \in \gamma \cap \mathbf{R}$. Then $2|x - z| \leq l(\gamma) \leq 2m$ and hence $x \geq m$. Now for integers $j \geq m^2$ we have

$$|\phi(j+1) - \phi(j)| \leq 1/m, \quad \text{so } \operatorname{dist}(x, \partial D) \leq 1/2m.$$

Hence

$$l(\alpha) \geq 1 \geq m \operatorname{dist}(x, \partial D)$$

for either component α of $\gamma \setminus \{x\}$.

5. Comments

We refer to [GM, HK, K, V] for the definitions of certain terminology used below.

A careful examination of our proof reveals that the essential ingredients are the 'local weak quasisymmetry' property described in 2.3(b) and the 'local uniformity' property expressed in 2.1(a). Consequently we obtain the following list of equivalent descriptions for certain domains; a detailed proof will be supplied in a forthcoming paper.

Corollary. *Suppose D is quasiconformally equivalent to a uniform domain. Then the following are equivalent.*

(a) *D is a locally uniform domain.*

(b) *D is a W_p^1-extension domain for all $p \geq 1$.*
(c) *D is a W_n^1-extension domain.*
(d) *There exist constants a, ρ so that D satisfies* (2).
(e) *There exists a homeomorphism f of D onto a uniform domain and constants $h > 1$, $k > 1$ and ρ such that*

$$\begin{cases} |f(u) - f(v)| \geq k|f(w) - f(v)| \text{ whenever } u, v, w \in D, \\ \text{and } \operatorname{diam}\{u, v, w\} \leq \rho, \ |u - v| \geq h|w - v|. \end{cases} \tag{4}$$

Remarks. (a) Notice that W_n^1-extension domains which are quasiconformally equivalent to uniform domains are in fact W_p^1-extension domains for all $p \geq 1$. Also, it turns out that (4) characterizes the quasiconformal homeomorphisms of locally uniform domains onto uniform domains.

(b) Condition (2) is a weak version of F. W. Gehring's *linear local connectivity* (LLC) [GM, p. 186]. Clearly our theorem remains valid when "D is a W_n^1-extension domain" is replaced by "D satisfies (2) for some a, ρ."

(c) Jones [J2] actually proved that 2.1(a) holds for (c, r)-locally uniform domains, however in this situation one obtains G only for $0 < t < r$. Thus 2.0 characterizes local uniformity.

(d) From our examples we observe that the hypothesis "D is quasiconformally equivalent to a uniform domain" in the corollary cannot be replaced, e.g., by "D is quasiconformally equivalent to a locally uniform domain," even if we replace any of (b), (c), (d) by the stronger conditions that D is an LLC, a QED, or an L_n^1-extension domain.

(e) Condition (4) is a local version of P. Tukia and J. Väisälä's notion of a *weak-quasisymmetry* [TV, p. 98]. In fact, (4) implies that f^{-1} is locally weakly quasisymmetric. However, there exist homeomorphisms which satisfy (4) whose inverses are not weakly quasisymmetric; e.g., map an infinite cylinder quasiconformally onto a half-space—by [TV, 2.16] and [V, 3.2, 4.11] such a quasiconformal homeomorphism cannot have a weakly quasisymmetric inverse.

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