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ASYMPTOTIC FORMULAS FOR ULTRASPHERICAL POLYNOMIALS $P_n^{(\lambda)}(x)$ AND THEIR ZEROS FOR LARGE VALUES OF λ

ÁRPÁD ELBERT AND ANDREA LAFORGIA

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ABSTRACT. For $\lambda > -1/2$ we denote by $P_n^{(\lambda)}(x)$ the ultraspherical polynomial of degree n and by $x_{n,k}^{(\lambda)}$ and $h_{n,k}$ $(k = 1, 2, \dots, n)$ the kth zeros of $P_n^{(\lambda)}(x)$ and of the Hermite polynomial $H_n(x)$, respectively. In this paper we establish the following formulas

$$\lambda^{-n/2} P_n^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right) = \sum_{j=0}^{n-1} \lambda^{-j} Q_{nj}(x) \quad \text{for } \lambda \neq 0$$

and

$$\begin{aligned} x_{n,k}^{(\lambda)} &= h_{n,k}\lambda^{-1/2} - \frac{h_{n,k}}{8}(2n - 1 + 2h_{n,k}^2)\lambda^{-3/2} \\ &\quad + h_{n,k}\left(\frac{12n^2 - 12n + 1}{128} + \frac{5n-2}{24}h_{n,k}^2 + \frac{5}{96}h_{n,k}^4\right)\lambda^{-5/2} \\ &\quad + O(\lambda^{-7/2}), \quad \lambda \to \infty \end{aligned}$$

where $Q_{n0}(x) = H_n(x)/n!$ and $Q_{nj}(x)$ $(j = 1, 2, \dots, n-1)$ are polynomials specified in Theorem 1. Finally we show that the positive (negative) zeros of $P_n^{(\lambda)}(x)$ are convex (concave) functions of λ, provided λ is sufficiently large.

1. INTRODUCTION AND RESULTS

In a private communication F. Calogero mentioned the desirability of obtaining an asymptotic formula for the zeros $x_{n,k}^{(\lambda)}$ $(k = 1, 2, \dots, n)$ of the ultraspherical polynomials $P_n^{(\lambda)}(x)$ as $\lambda \to \infty$. Clearly the known limit relation [2, p. 107]

$$\lim_{\lambda\to\infty} \lambda^{-n/2} P_n^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right) = \frac{H_n(x)}{n!},$$

where $H_n(x)$ denotes the Hermite polynomial, implies

$$\lim_{\lambda\to\infty} \sqrt{\lambda}\, x_{n,k}^{(\lambda)} = h_{n,k}, \qquad k = 1, 2, \dots, n$$

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where $h = h_{n,k}$ denotes the kth zero of $H_n(x)$. Following the notations in [2] the zeros $x_{n,k}$ and $h_{n,k}$ $(k = 1, 2, \ldots, n)$ are in decreasing order.

In this paper we establish the first three terms of the asymptotic expansion for $x_{n,k}^{(\lambda)}$ as $\lambda \to \infty$. In order to do this we need an expansion of $P_n^{(\lambda)}(x)$ as a polynomial of $1/\lambda$ for $\lambda \neq 0$.

Theorem 1. *The polynomial $P_n^{(\lambda)}(x)$ has the representation*

$$\lambda^{-n/2} P_n^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right) = \sum_{j=0}^{n-1} \lambda^{-j} Q_{nj}(x) \quad \text{for } \lambda \neq 0 \tag{1.1}$$

where

$$Q_{nj}(x) = \sum_{m=0}^{\min([n/2], n-j-1)} (-1)^{m+j} S_{n-m}^{(n-m-j)} \frac{(2x)^{n-2m}}{m!(n-2m)!} \tag{1.2}$$

and $S_i^{(l)}$ are the Stirling numbers of the first kind (see [1], *p.* 840), *particularly*

$$S_i^{(i)} = 1, \qquad S_i^{(i-1)} = -\frac{i(i-1)}{2}, \qquad S_i^{(i-2)} = \frac{1}{8}i^4 - \frac{5}{12}i^3 + \frac{3}{8}i^2 - \frac{1}{12}i, \tag{1.3}$$

and we observe here that $Q_{n0}(x) = H_n(\lambda)/n!$.

Using the expansion above we derive the asymptotic formula of $x_{n,k}^{(\lambda)}$.

Theorem 2. *For the zeros $x_{n,k}^{(\lambda)}$ of $P_n^{(\lambda)}(x)$ the following asymptotic formula*

$$\begin{aligned} x_{n,k}^{(\lambda)} &= h_{n,k}\lambda^{-1/2} - \frac{h_{n,k}}{8}(2n - 1 + 2h_{n,k}^2)\lambda^{-3/2} \\ &\quad + h_{n,k}\left(\frac{12n^2 - 12n + 1}{128} + \frac{5n-2}{24}h_{n,k}^2 + \frac{5}{96}h_{n,k}^4\right)\lambda^{-5/2} \\ &\quad + O(\lambda^{-7/2}), \qquad \lambda \to \infty \end{aligned} \tag{1.4}$$

holds.

We are also concerned here with the *convexity* (*concavity*) property with respect to λ of the positive (negative) zeros $x_{n,k}^{(\lambda)}$ of $P_n^{(\lambda)}(x)$. This property follows directly from the following result.

Theorem 3. *For $x_{n,k}^{(\lambda)}$ $(k = 1, 2, \ldots, n)$ the limit relations*

$$\lim_{\lambda\to\infty} \lambda^{3/2} \frac{\partial}{\partial\lambda} x_{n,k}^{(\lambda)} = -\frac{1}{2} h_{n,k}$$

$$\lim_{\lambda\to\infty} \lambda^{5/2} \frac{\partial^2}{\partial\lambda^2} x_{n,k}^{(\lambda)} = \frac{3}{4} h_{n,k}$$

hold.

The proofs of the above theorems are given in the next section.

2. The proofs

Proof of Theorem 1. The ultraspherical polynomial $P_n^{(\lambda)}(x)$ admits the following explicit representation [2, p. 84]

$$P_n^{(\lambda)}(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{\Gamma(n-m+\lambda)}{\Gamma(\lambda)\Gamma(m+1)\Gamma(n-2m+1)} (2x)^{n-2m} \tag{2.1}$$

where Γ denotes the gamma function.

Recalling the definition of the Stirling numbers [1, p. 840] we have

$$(x)_i = x(x-1)\cdots(x-i+1) = \sum_{i=1}^{i} S_i^{(l)} x^l ,$$

hence

$$\frac{\Gamma(i+\lambda)}{\Gamma(\lambda)} = \lambda(\lambda+1)\cdots(\lambda+i-1) = (-1)^i(-\lambda)_i = \sum_{l=1}^{i}(-1)^{(i-l)} S_i^{(l)} \lambda^l. \tag{2.2}$$

The particular values of $S_i^{(i)}$, $S_i^{(i-1)}$, $S_i^{(i-2)}$ given in (1.3) can be taken from the asymptotic expansion of $\Gamma(i+\lambda)/\Gamma(\lambda)$ in [1, p. 257; 6.1.47].

Putting (2.2) in (2.1) we get

$$P_n^{(\lambda)}(x) = \sum_{m=0}^{[n/2]}(-1)^m \sum_{l=1}^{n-m}(-1)^{n-m-l} S_{n-m}^{(l)} \lambda^l \frac{(2x)^{n-2m}}{m!(n-2m)!} \tag{2.3}$$

and finally replacing x in (2.1) by $x/\sqrt{\lambda}$ we obtain the desired result.

Proof of Theorem 2. Let us look for the zeros $x_{n,k}^{(\lambda)}$ as

$$\lambda^{1/2} x_{n,k}^{(\lambda)} = h_{n,k} + \frac{\alpha}{\lambda} + \frac{\beta}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right), \qquad \lambda \to \infty.$$

For the sake of brevity we shall use the notations

$$Q_{n0}(x) = Q(x) \qquad Q_{n1}(x) = R(x) \qquad Q_{n2}(x) = S(x)$$

From (1.1) we get

$$\begin{aligned}
0 &= Q\left(h + \frac{\alpha}{\lambda} + \frac{\beta}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right)\right) + \frac{1}{\lambda} R\left(h + \frac{\alpha}{\lambda} + \frac{\beta}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right)\right) \\
&\quad + \frac{1}{\lambda^2} S\left(h + \frac{\alpha}{\lambda} + \frac{\beta}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right)\right) + O\left(\frac{1}{\lambda^3}\right) \\
&= \left(\frac{\alpha}{\lambda} + \frac{\beta}{\lambda^2}\right) Q'(h) + \frac{1}{2}\left(\frac{\alpha}{\lambda} + \frac{\beta}{\lambda^2}\right)^2 Q''(h) \\
&\quad + \frac{1}{\lambda}\left[R(h) + \frac{\alpha}{\lambda} R'(h) + \frac{1}{\lambda^2} S(h)\right] + O\left(\frac{1}{\lambda^3}\right) \\
&= \frac{1}{\lambda}[\alpha Q'(h) + R(h)] + \frac{1}{\lambda^2}\left[\beta Q'(h) + \frac{1}{2}\alpha^2 Q''(h) + \alpha R'(h) + S(h)\right] \\
&\quad + O\left(\frac{1}{\lambda^3}\right), \qquad \lambda \to \infty
\end{aligned}$$

where $h = h_{n,k}$. This implies that the coefficients of $1/\lambda$ and $1/\lambda^2$ on the right-hand side are zero. Therefore we have

$$\alpha = -\frac{R(h)}{Q'(h)}$$

and

$$\beta = -\frac{1}{2}\alpha^2 \frac{Q''(h)}{Q'(h)} - \alpha \frac{R'(h)}{Q'(h)} - \frac{S(h)}{Q'(h)}.$$

In the above formulas $Q'(h) \neq 0$ because $Q(h) = 0$ and $Q(x)$ is a solution of differential equation of Hermite polynomials [2, p. 106]

$$Q''(x) - 2xQ'(x) + 2nQ(x) = 0. \tag{2.4}$$

Another way to show $Q'(h) \neq 0$ could be the well-known fact that the orthogonal polynomials have only simple real zeros [2, p. 44].

To calculate α and β we need the values of $R(h)$, $R'(h)$, $S(h)$ and to this end we use the identities

$$\binom{i}{2} = \frac{n(n-2)}{8} + \frac{2n-1}{8}(2i-n) + \frac{1}{8}(2i-n)(2i-n-1) \tag{2.5}$$

$$\begin{aligned} b_i = {} & A(n) + B(n)(2i-n) + C(n)(2i-n)(2i-n-1) \\ & + D(n)(2i-n)(2i-n-1)(2i-n-2) \\ & + E(n)(2i-n)(2i-n-1)(2i-n-2)(2i-n-3), \end{aligned}$$

where

$$\begin{aligned} A(n) &= \frac{1}{3\cdot 128}(3n^4 - 20n^3 + 36n^2 - 16n) \\ B(n) &= \frac{1}{128}(4n^3 - 14n^2 + 8n + 1) \\ C(n) &= \frac{1}{128}(6n^2 - 8n - 1) \\ D(n) &= \frac{1}{3\cdot 64}(6n - 1) \\ E(n) &= \frac{1}{128}. \end{aligned}$$

Combining (2.5) and the formula of $R(x)$ in (1.2), we obtain

(2.6)

$$\begin{aligned} R(x) &= \sum_{m=0}^{[n/2]} (-1)^m \frac{\frac{n(n-2)}{8} + \frac{2n-1}{8}(n-2m) + \frac{1}{8}(n-2m)(n-2m-1)}{m!(n-2m)!} \\ & \qquad \times (2x)^{n-2m} \\ &= \frac{n(n-2)}{8}Q(x) + \frac{2n-1}{8}xQ'(x) + \frac{1}{8}x^2Q''(x). \end{aligned}$$

Similarly by (2.5)

$$\begin{aligned} S(x) = {} & A(n)Q(x) + B(n)xQ'(x) + C(n)x^2Q''(x) \\ & + D(n)x^3Q'''(x) + E(n)x^4Q^{IV}(x). \end{aligned} \tag{2.7}$$

From differential equation (2.4) we have

$$\begin{aligned} Q''(x) &= 2xQ'(x) - 2nQ(x) \\ Q'''(x) &= (4x^2 - 2n + 2)Q'(x) - 4nxQ(x) \\ Q^{IV}(x) &= 4x(2x^2 + 3 - 2n)Q'(x) - 4nQ(x)(2x^2 - n + 2). \end{aligned} \tag{2.8}$$

Thus $R(x)$ in (2.6) and $S(x)$ in (2.7) are linear combinations of $Q(x)$ and $Q'(x)$.

Differentiation of $R(x)$ in (2.6) shows that

$$R'(h) = \frac{n^2-1}{8}Q'(h) + \frac{2n+1}{8}hQ''(h) + \frac{h^2}{8}Q'''(h)$$

and finally we obtain

$$\alpha = -\frac{h}{8}(2n - 1 + 2h^2)$$
$$\beta = \frac{12n^2 - 12n - 1}{128}h + \frac{5n-2}{24}h^3 + \frac{5}{96}h^5.$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Since $P_n^{(\lambda)}(x_{n,k}^{(\lambda)}) \equiv 0$, differentiating with respect to λ we find

$$P_\lambda + P'x_\lambda = 0 \tag{2.9}$$

where

$$P_\lambda = \frac{\partial}{\partial\lambda}P_n^{(\lambda)}(x)|_{x=x_{n,k}^{(\lambda)}}, \qquad P' = \frac{\partial}{\partial x}P_n^{(\lambda)}(x)|_{x=x_{n,k}^{(\lambda)}}, \qquad x_\lambda = \frac{\partial}{\partial\lambda}x_{n,k}^{(\lambda)}.$$

Hence

$$x_\lambda = -\frac{P_\lambda}{P'}.$$

Clearly $P' \neq 0$ because $P_n^{(\lambda)}(x)$ is a solution of the second-order differential equation [2, p. 80]

$$(1-x^2)y'' - (2\lambda+1)xy' + n(n+2\lambda)y = 0. \tag{2.10}$$

Differentiation with respect to λ of (2.9) gives

$$\frac{\partial^2}{\partial\lambda^2}x_{n,k}^{(\lambda)} = x_{\lambda\lambda} = -\frac{P_{\lambda\lambda}P' - P_\lambda P'_\lambda}{[P']^2} - \frac{P_\lambda P' - P_\lambda P''}{[P']^2}x_\lambda \tag{2.11}$$

where

$$P_{\lambda\lambda} = \frac{\partial^2}{\partial\lambda^2}P_n^{(\lambda)}(x)|_{x=x_{n,k}^{(\lambda)}}.$$

From (2.3) we get

$$\frac{d}{dx}P_n^{(\lambda)}(x) = \sum_{m=0}^{[n/2]}(-1)^m\frac{\lambda^{n-m}+\dots}{m!(n-2m)!}(n-2m)\cdot 2\cdot(2x)^{n-2m-1}$$

from which

$$\lim_{\lambda\to\infty}\lambda^{-(n+1)/2}\frac{d}{dx}P_n^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right) = Q'(x) \tag{2.12}$$

and

$$\frac{\partial}{\partial\lambda}P_n^{(\lambda)}(x) = \sum_{m=0}^{[n/2]}(-1)^m\frac{(n-m)\lambda^{n-m-1}+\dots}{m!(n-2m)!}(2x)^{n-2m}. \tag{2.13}$$

Since $n-m=n/2+(n-2m)/2$ we obtain the limit relation

$$\lim_{\lambda\to\infty}\lambda^{-(n-2)/2}\frac{\partial}{\partial\lambda}P_n^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right)=\sum_{m=0}^{[n/2]}(-1)^m\frac{\frac{n}{2}+\frac{n-2m}{2}}{m!(n-2m)!}(2x)^{n-2m} \tag{2.14}$$
$$=\frac{n}{2}Q(x)+\frac{1}{2}xQ'(x).$$

Using (2.12) and (2.13) in (2.9), by Theorem 2 we get

$$\lim_{\lambda\to\infty}\lambda^{3/2}x_\lambda=-\frac{1}{2}h \tag{2.15}$$

which gives the first part of Theorem 3.

To prove the second part we consider formula (2.13).

Differentiation with respect to x and with respect to λ gives

$$\frac{\partial^2}{\partial\lambda\partial x}P_n^{(\lambda)}(x)=\sum_{m=0}^{[n/2]}(-1)^m\frac{(n-m)\lambda^{n-m-1}+\dots}{m!(n-2m)!}(n-2m)\cdot 2\cdot(2x)^{n-2m-1}$$

and

$$\frac{\partial^2}{\partial\lambda^2}P_n^{(\lambda)}(x)=\sum_{m=0}^{[n/2]}(-1)^m\frac{(n-m)(n-m-1)\lambda^{n-m-2}+\dots}{m!(n-2m)!}(2x)^{n-2m}$$

respectively.

Replacing x by $x/\sqrt{\lambda}$ in the last two relations we find

(2.16)

$$\lim_{\lambda\to\infty}\lambda^{-(n-1)/2}\frac{\partial^2}{\partial\lambda\partial x}P_n^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right)$$
$$=\sum_{m=0}^{[n/2]}(-1)^m\frac{(n-m)(n-2m)}{m!(n-2m)!}\cdot 2\cdot(2x)^{n-2m-1}=\frac{n+1}{2}Q'(x)+\frac{1}{2}xQ''(x)$$

and

$$\lim_{\lambda\to\infty}\lambda^{-(n-4)/2}\frac{\partial^2}{\partial\lambda^2}P_n^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right)$$
$$=\sum_{m=0}^{[n/2]}(-1)^m\frac{(n-m)(n-m-1)}{m!(n-2m)!}\cdot(2x)^{n-2m}=2R(x) \tag{2.17}$$

where $R(x)$ has been defined in (1.2).

Now by differential equation (2.10), taking into consideration (2.12) we have also

$$\lim_{\lambda\to\infty}\lambda^{-(n+2)/2}\frac{d^2}{dx^2}P_n^{(\lambda)}\left(\frac{h}{\sqrt{\lambda}}\right)=2hQ'(h). \tag{2.18}$$

Finally using limit relations (2.12), (2.14), (2.15), (2.16), (2.17), and (2.18) in (2.11), by means of straightforward calculations we are led to

$$\lim_{\lambda\to\infty}\lambda^{5/2}x_{\lambda\lambda}=-\left[\frac{2R(h)}{Q'(h)}-\frac{h}{2}\left(\frac{n+1}{2}+\frac{h}{2}\frac{Q''(h)}{Q'(h)}\right)\right]+\left[\frac{n+1}{2}+\frac{h}{2}\frac{Q''(h)}{Q'(h)}-h^2\right]\frac{h}{2}.$$

Now by (2.8)

$$\frac{Q''(h)}{Q'(h)} = 2h, \qquad \frac{R(h)}{Q'(h)} = \frac{1}{8}[(2n-1)h + 2h^3]$$

and therefore

$$\lim_{\lambda\to\infty} \lambda^{5/2} x_{\lambda\lambda} = \frac{3}{4}h.$$

The proof of Theorem 3 is complete.

Remark 1. The known formula [2, p. 132]

$$h_{n,k} = \sqrt{2n+1} - 6^{-1/3}(2n+1)^{-1/6}(i_k + \varepsilon_n), \qquad \varepsilon_n \to 0 \text{ as } n \to \infty$$

where i_k is the kth positive zero of the Airy's function of the first kind shows that in (1.4) $h_{n,k}$ can be replaced by

$$\sqrt{2n+1} - 6^{-1/3}(2n+1)^{-1/6} i_k.$$

In this way we need only to know the values of i_k for which there are extended tables.

Remark 2. Theorem 2 gives sharp numerical results. For example, when $n = 2$ and $\lambda = 100$ we find $h_{2,1} = 1/\sqrt{2}$. Using this result in (1.4) we get $\sqrt{2}x_{n,1}^{(100)} = 0.09950375\ldots$ while the exact value is $\sqrt{2}x_{n,1}^{(100)} = 1/\sqrt{101} = 0.09950371\ldots$.

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Mathematical Institute of the Hungarian Academy of Sciences, Budapest, P.O.B. 127, H-1364, Hungary

Facoltà d'Ingegneria, Università degli Studi dell'Aquila, Monteluco di Roio, 67040 L'Aquila, Italy