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UNIFORM CONVERGENCE OF ERGODIC LIMITS AND APPROXIMATE SOLUTIONS

SEN-YEN SHAW

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ABSTRACT. Let A be a densely defined closed (linear) operator, and $\{A_\alpha\}$, $\{B_\alpha\}$ be two nets of bounded operators on a Banach space X such that $\|A_\alpha\| = O(1)$, $A_\alpha A \subset AA_\alpha$, $\|AA_\alpha\| = o(1)$, and $B_\alpha A \subset AB_\alpha = I - A_\alpha$. Denote the domain, range, and null space of an operator T by $D(T)$, $R(T)$, and $N(T)$, respectively, and let P (resp. B) be the operator defined by $Px = \lim_\alpha A_\alpha x$ (resp. $By = \lim_\alpha B_\alpha y$) for all those $x \in X$ (resp. $y \in \overline{R(A)}$) for which the limit exists. It is shown in a previous paper that $D(P) = N(A) \oplus \overline{R(A)}$, $R(P) = N(A)$, $D(B) = A(D(A) \cap \overline{R(A)})$, $R(B) = D(A) \cap \overline{R(A)}$, and that B sends each $y \in D(B)$ to the unique solution of $Ax = y$ in $\overline{R(A)}$. In this paper, we prove that $D(P) = X$ and $\|A_\alpha - P\| \to 0$ if and only if $\|B_\alpha|D(B) - B\| \to 0$, if and only if $\|B_\alpha|D(B)\| = O(1)$, if and only if $R(A)$ is closed. Moreover, when X is a Grothendieck space with the Dunford-Pettis property, all these conditions are equivalent to the mere condition that $D(P) = X$. The general result is then used to deduce uniform ergodic theorems for n-times integrated semigroups, (Y)-semigroups, and cosine operator functions.

1. INTRODUCTION

Let X be a Banach space and $B(X)$ be the set of all bounded linear operators on X. Let $A: D(A) \subset X \to X$ be a densely defined closed linear operator, and let $\{A_\alpha\}$ and $\{B_\alpha\}$ be two nets in $B(X)$ satisfying the conditions:

(C1) $\|A_\alpha\| = O(1)$, i.e., there exist M and α_0 such that $\|A_\alpha\| \le M$ for $\alpha \ge \alpha_0$,

(C2) $R(B_\alpha) \subset D(A)$ and $B_\alpha A \subset AB_\alpha = I - A_\alpha$ for all α,

(C3) $R(A_\alpha) \subset D(A)$ and $A_\alpha A \subset AA_\alpha$, and $\|AA_\alpha\| \to 0$.

These two systems of operators have been employed in our earlier papers [13] and [14] to formulate an abstract mean ergodic theorem and to produce approximate solutions of the functional equation Ax $= y$.

Let P be the operator defined by $Px := s\text{-}\lim_\alpha A_\alpha x$ for $x \in D(P) := \{x \in X; s\text{-}\lim_\alpha A_\alpha x \text{ exists}\}$, and let B be the operator defined by $By := s\text{-}\lim_\alpha B_\alpha y$ for $y \in D(B) := \{y \in \overline{R(A)}; s\text{-}\lim_\alpha B_\alpha y \text{ exists}\}$. The following two

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strong convergence theorems were proved in [13]: (i) P is a bounded linear projection with range $R(P) = N(A)$, null space $N(P) = \overline{R(A)}$, and domain $D(P) = N(A) \oplus \overline{R(A)} = \{x \in X; \{A_\alpha x\}$ has a weak cluster point$\}$; (ii) B is the inverse operator of the restriction $A|\overline{R(A)}$ of A to $\overline{R(A)}$; it has range $R(B) = D(A) \cap \overline{R(A)}$ and domain $D(B) = A(D(A) \cap \overline{R(A)}) = \{y \in \overline{R(A)}; \{B_\alpha y\}$ has a weak cluster point$\}$. Moreover, for any given $y \in D(B)$, the vector By is the unique solution of the functional equation $\mathrm{Ax} = y$ that lies in $\overline{R(A)}$. This closed operator B is called the inner inverse of A.

Actually, the convergence of $B_\alpha y$ to $By = (A|\overline{R(A)})^{-1}y$ for $y \in D(B)$ is seen from the following computation using (C2) and (i).

$$(*)\qquad \begin{aligned}\|B_\alpha y - By\| &= \|B_\alpha ABy - By\| = \|(B_\alpha A - I)By\| \\ &= \|A_\alpha By\| = \|(A_\alpha - P)By\| \to 0.\end{aligned}$$

$\{A_\alpha\}$ is said to be strongly ergodic if $D(P) = X$. In this case, we have

$$R(A) = A(D(A) \cap X) = A(D(A) \cap [N(A) \oplus \overline{R(A)}]) = A(D(A) \cap \overline{R(A)}) = D(B).$$

Conversely, the equality $D(B) = R(A)$ implies the strong ergodicity because, if not, there would be a $z \in D(A) \setminus D(P)$ and an $x \in D(A) \cap \overline{R(A)}$ such that $Az = \mathrm{Ax}$, which leads to $z = (z - x) + x \in N(A) \oplus \overline{R(A)} = D(P)$, a contradiction.

The purpose of this paper is to prove the following two uniform convergence theorems for the two systems $\{A_\alpha\}$ and $\{B_\alpha\}$. Applications to concrete examples are to be given in §3.

Theorem 1. *Let A be a densely defined closed linear operator in X, and let $\{A_\alpha\}$ and $\{B_\alpha\}$ be two nets in $B(X)$ which satisfy* (C1), (C2), *and* (C3).

Then the following statements are equivalent:

(1) $\|A_\alpha|D(P) - P\| \to 0$,
(2) $D(P) = X$ *and* $\|A_\alpha - P\| \to 0$,
(3) $R(A)$ *is closed,*
(4) $R(A^2)$ *is closed,*
(5) $X = N(A) \oplus R(A)$,
(6) $\|B_\alpha|R(A)\| = O(1)$,
(7) B *is bounded and* $\|B_\alpha|D(B) - B\| \to 0$.

Moreover, if (1)–(7) *hold, then* $D(B) = R(A^2) = R(A)$, $\|A_\alpha - P\| \le (M+1) \times \|AA_\alpha\|\,\|B\|$ *and* $\|B_\alpha|D(B) - B\| \le (M+1)\|AA_\alpha\|\,\|B\|^2$.

A Banach space X is called a Grothendieck space if every weakly* convergent sequence in the dual space X^* is weakly convergent, and is said to have the Dunford-Pettis property if $\langle x_n, x_n^*\rangle \to 0$ whenever $x_n \to 0$ weakly in X and $x_n^* \to 0$ weakly in X^*. Among examples of Grothendieck spaces with the Dunford-Pettis property are L^∞, $B(s, \Sigma)$, $H^\infty(D)$, etc. (See [7].)

Theorem 2. *Let X be a Grothendieck space with the Dunford-Pettis property, and let A, $\{A_\alpha\}$, and $\{B_\alpha\}$ be as in Theorem* 1. *If $\{A_\alpha\}$ is strongly ergodic, then it is uniformly ergodic.*

Thus, in this case the conditions (1)–(7) all are equivalent to each of $D(P) = X$ and $D(B) = R(A)$. In view of the next theorem, we have another equivalent condition: $\overline{R(A^*)} = w^* - \mathrm{cl}\,(R(A^*))$.

Theorem 3. *Let X be a Grothendieck space, and let A, $\{A_\alpha\}$, and $\{B_\alpha\}$ be as in Theorem* 1. *Then $\{A_\alpha\}$ is strongly ergodic if and only if $\overline{R(A^*)} = w^* - \mathrm{cl}\,(R(A^*))$.*

2. Proof of Main Result

Proof of Theorem 1. We prove the implication s: $(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2)$, $(1) \Rightarrow (3) \Rightarrow (6) \Rightarrow (1) + (7)$.

$(2) \Rightarrow (3)$ (C3) implies that $\overline{R(A)}$ is invariant under A_α and $PA = 0$, so that $\|A_\alpha|\overline{R(A)}\| = \|(A_\alpha - P)|\overline{R(A)}\| \leq \|A_\alpha - P\| \to 0$. Hence for some α, $(A_\alpha - I)|\overline{R(A)}$ is invertible and so $\overline{R(A)} \subset R(A_\alpha - I) \subset R(A)$.

$(3) \Rightarrow (4)$. By the open mapping theorem (cf. [15, p. 213]), there is some $m > 0$ such that each $x \in R(A)$ is equal to Ay for some $y \in D(A)$ with $\|y\| \leq m\|x\|$. Hence $\|A_\alpha x\| = \|AA_\alpha y\| \leq \|AA_\alpha\| m\|x\|$, showing that $\|A_\alpha|R(A)\| \leq m\|AA_\alpha\| \to 0$ and so $(A_\alpha - I)|R(A)$ is invertible for some α. This, together with (C2) and (C3), implies that $R(A) = (A_\alpha - I)AD(A) = A(A_\alpha - I)D(A) \subset A(D(A) \cap R(A)) = R(A^2)$. Hence $R(A^2) = R(A)$ and is closed.

$(4) \Rightarrow (5)$. If $x \in D(A)$, then Ax $= \lim\{AA_\alpha x - A(A_\alpha - I)x\} = -\lim A(A_\alpha - I)x \in \overline{R(A^2)} = R(A^2)$. This shows that $R(A) = R(A^2)$ is closed and $D(A) \subset N(A) + R(A)$. Next, let $x \in X$, and let $\{x_n\}$ be a sequence in $D(A)$ such that $x_n \to x$. Then $A_\alpha x \in D(A)$ and $(A_\alpha - I)x = \lim_{n\to\infty}(A_\alpha - I)x_n \in \overline{R(A)} = R(A)$ so that $x = A_\alpha x - (A_\alpha - I)x \in D(A) + R(A) \subset N(A) + R(A)$. Hence $X = N(A) + R(A)$. To see that this is a direct sum, let $x \in N(A) \cap R(A)$. Then there is a y such that $Ay = x \in N(A) \subset N(A_\alpha - I)$ for all α. But then $x = A_\alpha x = A_\alpha A y \to 0$.

$(5) \Rightarrow (2)$. The closedness of A and assumption (5) imply $R(A)$ is closed (see [16, p. 217]). Then, as was shown in $(3) \Rightarrow (4)$, we have $\|(A_\alpha - P)|R(A)\| = \|A_\alpha|R(A)\| \leq m\|AA_\alpha\| \to 0$. Because $(A_\alpha - P)|N(A) = 0$, it follows that $\|A_\alpha - P\| \to 0$.

$(1) \Rightarrow (3)$. Since $A_\alpha A \subset AA_\alpha$, the space $D(P) = N(A) \oplus \overline{R(A)}$ is invariant under A_α, and $A|D(P)$ is a densely defined closed operator in $D(P)$. Applying $(2) \Rightarrow (5)$ to $\{A_\alpha|D(P)\}$ and $A|D(P)$, we infer that $D(P) = N(A|D(P)) \oplus R(A|D(P))$. Since $N(A|D(P)) = N(A)$ and $R(A|D(P)) \subset R(A)$, it follows from the two expressions of $D(P)$ that $\overline{R(A)} = R(A|D(P))$ and hence $\overline{R(A)} = R(A)$.

$(3) \Rightarrow (6)$. If $R(A)$ is closed, then as shown in $(3) \Rightarrow (4)$, we have $R(A^2) = R(A)$, so that $D(B) = A(D(A) \cap R(A)) = R(A^2) = R(A)$ is closed. Since $B_\alpha y \to By$ for all $y \in D(B)$, the uniform boundedness principle implies (6).

$(6) \Rightarrow (1) + (7)$. (6) implies that B is bounded and hence $R(A|D(P)) = A(D(A) \cap \overline{R(A)}) = D(B)$ is closed. An application of $(3) \Rightarrow (2)$ to $\{A_\alpha|D(P)\}$ asserts that $\|A_\alpha|D(P) - P\| \to 0$, from which, together with $(*)$, we see that $\|B_\alpha|D(B) - B\| \leq \|A_\alpha|D(P) - P\|\,\|B\| \to 0$.

Finally, if (1)–(7) hold, then for any $x \in X$, we have $x - Px \in R(A) = R(A^2) = D(B)$ so that $AB(x - Px) = x - Px$ and

$$\|A_\alpha x - Px\| = \|A_\alpha(x - Px)\| = \|A_\alpha AB(I - P)x\| \leq \|AA_\alpha\|\,\|B\|(M + 1)\|x\|.$$

Hence $\|A_\alpha - P\| \leq (M+1)\|AA_\alpha\|\,\|B\|$ and $\|B_\alpha|D(B) - B\| \leq (M+1)\|AA_\alpha\|\,\|B\|^2$.

To prove Theorems 2 and 3 we need the following lemma.

Lemma. *If* A, $\{A_\alpha\}$, *and* $\{B_\alpha\}$ *satisfy conditions* (C1), (C2), *and* (C3), *then* A^*, $\{A_\alpha^*\}$, *and* $\{B_\alpha^*\}$ *also satisfy these conditions.*

Proof. It suffices to show that if $E \in B(X)$ is such that $R(E) \subset D(A)$ and $EA \subset AE$, then $R(E^*) \subset D(A^*)$ and $E^*A^* \subset A^*E^* = (AE)^*$. If $x^* \in D(A^*)$, then $\langle \mathrm{Ax}, E^*x^*\rangle = \langle E\,\mathrm{Ax}, x^*\rangle = \langle x, E^*A^*x^*\rangle$ for all $x \in D(A)$, so that $E^*x^* \in D(A^*)$ and $A^*E^*x^* = E^*A^*x^* = (AE)^*x^*$. Hence $E^*A^* \subset A^*E^*$. To show $R(E^*) \subset D(A^*)$ and $A^*E^* = (AE)^*$, we use the fact that A^* is weakly* densely defined and $w^* - w^*$-closed. For any $x^* \in X^*$, let $\{x_\alpha^*\}$ be a net in $D(A^*)$ such that $x_\alpha^* \to x^*$ weakly*. Then $E^*x_\alpha^*$ and $A^*E^*x_\alpha^*$ $(= (AE)^*x_\alpha^*)$ converges weakly* to E^*x^* and $(AE)^*x^*$, respectively. This implies that $E^*x^* \in D(A^*)$ and $A^*E^*x^* = (AE)^*x^*$.

Proof of Theorem 2. Since $A_\alpha|R(P) = I$, we may assume $A_\alpha \to 0$ strongly and show that $\|A_\alpha\| \to 0$, without loss of generality. Take a sequence $A_n \equiv A_{\alpha_n} \to 0$. Then A_nx converges strongly to 0 for all $x \in X$. This implies that $\{A_n^*x_n^*\}$ converges weakly* and hence weakly to 0 whenever $\{x_n^*\}$ is bounded. In particular, $\{A_n^*x^*\} \to 0$ weakly for all $x^* \in X^*$. The convergence actually holds in the strong topology, by the strong ergodic theorem (applied to $\{A_n^*\}$). This fact in turn implies that $\{A_nx_n\}$ converges weakly to 0 whenever $\{x_n\}$ is bounded. Now, it follows from a lemma of Lotz [7] that $\|A_n^2\| \to 0$. Thus $I - A_m$ is invertible for a sufficiently large m. By (C2) and (C3) we obtain that

$$\begin{aligned}\|A_n\| &= \|A_n(I - A_m)(I - A_m)^{-1}\| = \|A_nAB_m(I - A_m)^{-1}\| \\ &\leq \|AA_n\|\,\|B_m\|\,\|(I - A_m)^{-1}\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Application of Theorem 1 to $\{A_n\}$ and $\{A_\alpha\}$ shows first that $R(A)$ is closed and then that $\|A_\alpha\| \to 0$.

Proof of Theorem 3. If $D(P) = X$, then for every $x^* \in X^*$ we have

$$w\text{-}\lim_{n\to\infty} A_{\alpha_n}^*x^* = w^*\text{-}\lim_{n\to\infty} A_{\alpha_n}^*x^* = P^*x^*,$$

where $\{A_{\alpha_n}\}$ is any subsequence of $\{A_\alpha\}$ The strong ergodic theorem applied to $\{A_{\alpha_n}^*\}$, shows that $*P^*x^* = s\text{-}\lim_{n\to\infty} A_{\alpha_n}^*x^*$ for all $x^* \in X^*$, $N(P^*) = \overline{R(A^*)}$. Hence $\overline{R(A^*)} = R(P)^\perp = N(A)^\perp = [{}^\perp R(A^*)]^\perp = w^*\text{-}\mathrm{cl}\,(R(A^*))$.

Conversely, if $\overline{R(A^*)} = w^*\text{-}\mathrm{cl}\,(R(A^*))$, then $D(P)^\perp = \{N(A) \oplus \overline{R(A)}\}^\perp = [{}^\perp R(A^*)]^\perp \cap \overline{R(A)}^\perp = w^*\text{-}\mathrm{cl}\,(R(A^*)) \cap N(A^*) = \overline{R(A^*)} \cap N(A^*) = \{0\}$, again following from the strong ergodic theorem applied to $\{A_\alpha^*\}$. Since $D(P)$ is closed, it must be equal to X.

3. Examples

We consider applications to n-times integrated semigroups, (Y)-semigroups, and cosine operator functions.

3.1. ***n*-times integrated semigroups.** Let n be a positive integer. A strongly continuous family $\{T(t)\,;\, t \geq 0\}$ in $B(X)$ is called an n-times integrated semigroup (see [1, 15]) if $T(0) = I$ and

$$T(t)T(s) = \frac{1}{(n-1)!}\left(\int_t^{t+s} (t+s-r)^{n-1}T(r)\,dr - \int_0^s (t+s-r)^{n-1}T(r)\,dr\right) \qquad (s, t \geq 0).$$

A C_0-semigroup is called an o-times integrated semigroup. It is known that the integrals over $[0, t]$, $t \geq 0$, of an n-times $(n \geq 0)$ integrated semigroup form an $(n+1)$-times integrated semigroup, but not conversely.

$T(\cdot)$ is called nondegenerate if $T(t)x = 0$ for all $t > 0$ implies $x = 0$. It is called exponentially bounded if there are $M \geq 0$, $w \in R$ such that $\|T(t)\| \leq Me^{wt}$ for all $t \geq 0$. If $T(\cdot)$ is nondegenerate and exponentially bounded, then there exists a unique closed operator A satisfying $(w, \infty) \subset \rho(A)$ and $(\lambda - A)^{-1}x = \int_0^\infty \lambda^n e^{-\lambda t} T(t)\, dt$ for $x \in X$ and $\lambda > w$. This operator is called the generator of $T(\cdot)$. It is not necessarily densely defined. We only consider the case when A is densely defined. This includes all C_0-semigroups.

It is known [1, Proposition 3.3] that $\int_0^t T(s)x\, ds \in D(A)$ and $A\int_0^t T(s)x\, ds = T(t)x = (t^n/n!)x$ for all $x \in X$, and $\int_0^t T(s)\,\mathrm{Ax}\; ds = T(t)x - (t^n/n!)x$ for all $x \in D(A)$. Since A is closed, taking integration gives that

$$\int_0^t T(s)x\, ds - (t^{n+1}/(n+1)!)x = \begin{cases} A\int_0^t\int_0^s T(u)d\, du\, ds & \text{for } x \in X, \\ \int_0^t\int_0^s T(u)\,\mathrm{Ax}\;\; du\, ds & \text{for } x \in D(A). \end{cases}$$

Let $A_t := (n+1)!t^{-n-1}\int_0^t T(s)\, ds$ and $B_t := -(n+1)!t^{-n-1}\int_0^t\int_0^s T(u)\, du\, ds$ for $t > 0$. Then $B_tA \subset AB_t = I - A_t$ and $A_tA \subset AA_t = (n+1)!T(t)/t^{n+1} - (n+1)I/t$. Suppose $\|T(t)\| = O(t^n)$ $(t \to \infty)$. Then A, $\{A_t\}$, and $\{B_t\}$ satisfy (C1), (C2), and (C3) as $t \to \infty$. On the other hand, the systems $\{\lambda(\lambda - A)^{-1}\}$, $\{-(\lambda - A)^{-1}\}$ clearly satisfy (C1), (C2), and (C3) as $\lambda \to 0$ too. Hence the strong ergodic theorem and the theorems in §1 are applicable to $\{A_t\}$ with $\{B_t\}$ and $\{\lambda(\lambda - A)^{-1}\}$ with $\{-(\lambda - A)^{-1}\}$, and the next two theorems follow immediately.

Theorem 4. *Let $\{T(t)\,;\, t \geq 0\}$ be a nondegenerate n-times integrated semigroup with generator A densely defined. Suppose $\|T(t)\| = O(t^n)$ $(t \to \infty)$. Let A_t and B_t be as previously defined. Then $s\text{-}\lim_{t\to\infty} A_tx$ and $s\text{-}\lim_{\lambda\to 0^+} \lambda(\lambda - A)^{-1}x$ exist and are equal if one of them exists, and the limits define a bounded linear projection P onto $N(A)$ along $\overline{R(A)}$. For $y \in \overline{R(A)}$, $s\text{-}\lim_{t\to\infty} B_ty$ and $s\text{-}\lim_{\lambda\to 0^+}(A - \lambda)^{-1}y$ exist and are equal if one of them exists, and the limits define an operator B which sends each $y \in A(D(A) \cap \overline{R(A)})$ to the unique solution $x = By$ of $\mathrm{Ax} = y$ in $\overline{R(A)}$.*

Theorem 5. *Under the hypothesis of Theorem* 4, *the following statements are equivalent*:

(1) $\|A_t - P\| \to 0$ *as* $t \to \infty$,
(2) $\|\lambda(\lambda - A)^{-1} - P\| \to 0$ *as* $t \to 0^+$,
(3) $R(A)$ *is closed,*
(4) $\|B_t|R(A)\| = O(1)$ $(t \to \infty)$,
(5) $\|B_t|R(A) - B\| \to 0$ *as* $t \to \infty$
(6) $\|(\lambda - A)^{-1}|R(A) - B\| \to 0$ *as* $\lambda \to 0^+$.

Moreover, when X is a Grothendieck space with the Dunford-Pettis property, $D(P) = X$ and $\overline{R(A^*)} = w^*\text{-}\mathrm{cl}\,(R(A^*))$ are two more equivalent conditions.

Remarks. (i) When (1)–(6) hold, we have $\|A_t - P\| = O(1/t)$, $\|B_t|R(A) - B\| = O(1/t)$ $(t \to \infty)$, and $\|\lambda(\lambda - A)^{-1} - P\| = O(\lambda)$, $\|(\lambda - A)^{-1}|R(A) - B\| = O(\lambda)$, $\lambda \to 0^+$.

(ii) In the case $n = 0$, Theorem 4 is well known (see [3, pp. 58–60] for the first part, and [4] for the second part), the equivalence of (1), (2), and (3) in Theorem 5 is proved in [6] (see also [10]), the equivalence of strong ergodicity and $\overline{R(A^*)} = w^*\text{-cl}(RA^*))$ in a Grothendieck space is proved in [9], and the equivalence of strong ergodicity and uniform ergodicity in a Grothendieck space with the Dunford-Pettis property is proved in [7]. The theorems with $n \geq 1$ are new.

3.2. (Y)-**semigroups.** Let Y be a closed subspace of X^* such that the canonical imbedding of X into Y^* is isometric. A semigroup $\{T(t);\ t \geq 0\}$ of operators on X is called a (Y)-semigroup (cf. [8, 11]) if Y is invariant under $T^*(t)$ for all $t \geq 0$ and $T(\cdot)x$ is $\sigma(X, Y)$-continuous on $[0, \infty)$ and locally $\sigma(X, Y)$-Pettis integrable for all $x \in X$. The generator A of $T(\cdot)$ is defined by $\mathrm{Ax} := \sigma(X, Y)\text{-}\lim_{t\to 0^+} t^{-1}(T(t) - I)x$. A C_0-semigroup on X is a (X^*)-semigroup, and its dual semigroup is a (X)-semigroup. The tensor product $T(t)$ of two C_0-semigroups e^{tA} and e^{-tB} on X is a (Y)-semigroup on $B(X)$ for some suitable subspace Y of $B(X)^*$; its generator is the operator $\Delta\colon C \to AC - CB$.

The strong convergence of ergodic limits of a (Y)-semigroup and that of approximate solutions of the corresponding equation $\mathrm{Ax} = y$ have been discussed in [13, Example VI]. The result is the same as Theorem 4 with $n = 0$. By applying Theorems 1 and 2 one can easily see that Theorem 5 with $n = 0$ holds for (Y)-semigroups too. Since $S(t) := \int_0^t T(s)\,ds$, $t \geq 0$, forms a once-integrated semigroup, we can apply Theorems 4 and 5 to $S(\cdot)$ to obtain ergodic theorems for $(C, 2)$-means of $T(\cdot)$; they are Theorems 4 and 5 with $A_t = 2t^{-2}\int_0^t\int_0^s T(u)\,du\,ds$ and $B_t = -2t^{-2}\int_0^t\int_0^s\int_0^u T(v)\,dv\,ds$.

3.3. **Cosine operator functions.** A strongly continuous family $\{C(t);\, t \in R\}$ in $B(X)$ is called a cosine operator function if $C(0) = I$ and $C(t+s) + C(t-s) = 2C(t)C(s)$, s, $t \in R$. The generator A, defined by $\mathrm{Ax} := C''(0)x$, is a densely defined closed operator.

For $t > 0$ let

$$A_t := 2t^{-2}\int_0^t\int_0^s C(u)\,du\,ds$$

and

$$B_t = -2t^{-2}\int_0^t\int_0^s\int_0^u\int_0^v C(w)\,dw\,dv\,du\,ds.$$

Then we have $B_tA \subset AB_t \subset I_t - A$ and $A_tA \subset AA_t = 2t^{-2}(C(t) - I)$. The strong convergence of A_tx and B_ty as $t \to \infty$ has been discussed in [13, Example VII]. We now deduce from Theorems 1 and 2 the following theorem about uniform convergence.

Theorem 6. *Suppose that* $\|\int_0^t\int_0^s C(u)\,du\,ds\| = O(t^2)$ $(t \to \infty)$ *and* $\|C(t)\| = o(t^2)$ $(t \to \infty)$. *Then, with* A_t *and* B_t *defined as above, the conclusion of Theorem* 5 *remains valid.*

Concluding remark

Our Theorems 1, 2, and 3 can also be used to deduce uniform ergodic theorems for discrete semigroups (cf. [5, 7]) and uniform ergodic theorems for pseudoresolvents (cf. [10, 12]).

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DEPARTMENT OF MATHEMATICS, NATIONAL CENTRAL UNIVERSITY, CHUNG-LI, TAIWAN, REPUBLIC OF CHINA