UNIFORM CONVERGENCE OF ERGODIC LIMITS
AND APPROXIMATE SOLUTIONS

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Abstract. Let $A$ be a densely defined closed (linear) operator, and \{$A_{\alpha}$\}, \{$B_{\alpha}$\} be two nets of bounded operators on a Banach space $X$ such that \|$A_{\alpha}$\| = $O(1)$, $A_{\alpha}A \subset AA_{\alpha}$, \|$A_{\alpha}A$\| = $o(1)$, and $B_{\alpha}A \subset AB_{\alpha} = I - A_{\alpha}$. Denote the domain, range, and null space of an operator $T$ by $D(T)$, $R(T)$, and $N(T)$, respectively, and let $P$ (resp. $B$) be the operator defined by $Px = \lim_{\alpha} A_{\alpha}x$ (resp. $By = \lim_{\alpha} B_{\alpha}y$) for all those $x \in X$ (resp. $y \in R(A)$) for which the limit exists. It is shown in a previous paper that $D(P) = N(A) \cap R(A)$, $R(P) = N(A)$, $D(B) = A(D(A) \cap R(A))$, $R(B) = D(A) \cap R(A)$, and that $B$ sends each $y \in D(B)$ to the unique solution of $Ax = y$ in $R(A)$. In this paper, we prove that $D(P) = X$ and $\|P - A\| \to 0$ if and only if $\|B_{\alpha}D(B) - B\| \to 0$, if and only if $\|B_{\alpha}D(B)\| = O(1)$, if and only if $R(A)$ is closed. Moreover, when $X$ is a Grothendieck space with the Dunford-Pettis property, all these conditions are equivalent to the mere condition that $D(P) = X$. The general result is then used to deduce uniform ergodic theorems for $n$-times integrated semigroups, $(Y)$-semigroups, and cosine operator functions.

1. Introduction

Let $X$ be a Banach space and $B(X)$ be the set of all bounded linear operators on $X$. Let $A\colon D(A) \subset X \to X$ be a densely defined closed linear operator, and let \{$A_{\alpha}$\} and \{$B_{\alpha}$\} be two nets in $B(X)$ satisfying the conditions:

(C1) $\|A_{\alpha}\| = O(1)$, i.e., there exist $M$ and $\alpha_0$ such that $\|A_{\alpha}\| \leq M$ for $\alpha \geq \alpha_0$,

(C2) $R(B_{\alpha}) \subset D(A)$ and $B_{\alpha}A \subset AB_{\alpha} = I - A_{\alpha}$ for all $\alpha$,

(C3) $R(A_{\alpha}) \subset D(A)$ and $A_{\alpha}A \subset AA_{\alpha}$, and $\|AA_{\alpha}\| \to 0$.

These two systems of operators have been employed in our earlier papers [13] and [14] to formulate an abstract mean ergodic theorem and to produce approximate solutions of the functional equation $Ax = y$.

Let $P$ be the operator defined by $Px := \lim_{\alpha} A_{\alpha}x$ for $x \in D(P) := \{x \in X; \lim_{\alpha} A_{\alpha}x \text{ exists}\}$, and let $B$ be the operator defined by $By := \lim_{\alpha} B_{\alpha}y$ for $y \in D(B) := \{y \in R(A); \lim_{\alpha} B_{\alpha}y \exists\}$. The following two
strong convergence theorems were proved in [13]: (i) \( P \) is a bounded linear projection with range \( R(P) = N(A) \), null space \( N(P) = R(A) \), and domain \( D(P) = N(A) \oplus R(A) = \{ x \in X ; \{ A_\alpha x \} \text{ has a weak cluster point} \} \); (ii) \( B \) is the inverse operator of the restriction \( A|R(A) \) of \( A \) to \( R(A) \); it has range \( R(B) = D(A) \cap R(A) \) and domain \( D(B) = A(D(A) \cap R(A)) = \{ y \in R(A) \}; \{ B_\alpha y \} \) has a weak cluster point}. Moreover, for any given \( y \in D(B) \), the vector \( By \) is the unique solution of the functional equation \( Ax = y \) that lies in \( R(A) \). This closed operator \( B \) is called the inner inverse of \( A \).

Actually, the convergence of \( B_\alpha y \) to \( By = (A|R(A))^{-1}y \) for \( y \in D(B) \) is seen from the following computation using \((C2)\) and \((i)\).

\[
\| B_\alpha y - By \| = \| B_\alpha ABy - By \| = \| B_\alpha A - I \| By \|
\]

\((*)\)

\[
\| B_\alpha ABy - B_\alpha y \| = \| (A_\alpha - P)By \| \to 0 .
\]

\( \{ A_\alpha \} \) is said to be strongly ergodic if \( D(P) = X \). In this case, we have \( R(A) = A(D(A) \cap X) = A(D(A) \cap [N(A) \oplus R(A)]) = A(D(A) \cap R(A)) = D(B) \).

Conversely, the equality \( D(B) = R(A) \) implies the strong ergodicity because, if not, there would be a \( z \in D(A) \setminus D(P) \) and an \( x \in D(A) \cap R(A) \) such that \( Az = Ax \), which leads to \( z = (z - x) + x \in N(A) \oplus R(A) = D(P) \), a contradiction.

The purpose of this paper is to prove the following two uniform convergence theorems for the two systems \( \{ A_\alpha \} \) and \( \{ B_\alpha \} \). Applications to concrete examples are to be given in §3.

**Theorem 1.** Let \( A \) be a densely defined closed linear operator in \( X \), and let \( \{ A_\alpha \} \) and \( \{ B_\alpha \} \) be two nets in \( B(X) \) which satisfy \((C1), (C2), \) and \((C3)\).

Then the following statements are equivalent:

1. \( \| A_\alpha \|D(P) - P\| \to 0 \),
2. \( D(P) = X \) and \( \| A_\alpha - P\| \to 0 \),
3. \( R(A) \) is closed,
4. \( R(A^2) \) is closed,
5. \( X = N(A) \oplus R(A) \),
6. \( \| B_\alpha \|\|R(A)\| = O(1) \),
7. \( B \) is bounded and \( \| B_\alpha \|D(B) - B\| \to 0 \).

Moreover, if \((1)\)-(7) hold, then \( D(B) = R(A^2) = R(A) \), \( \| A_\alpha - P \| \leq (M + 1) \times \| A_\alpha \| \| B \| \) and \( \| B_\alpha \|D(B) - B\| \leq (M + 1) \| A_\alpha \| \| B \| ^2 \).

A Banach space \( X \) is called a Grothendieck space if every weakly* convergent sequence in the dual space \( X^* \) is weakly convergent, and is said to have the Dunford-Pettis property if \( \langle x_n, x_n^* \rangle \to 0 \) whenever \( x_n \to 0 \) weakly in \( X \) and \( x_n^* \to 0 \) weakly in \( X^* \). Among examples of Grothendieck spaces with the Dunford-Pettis property are \( L^\infty \), \( B(s, \Sigma) \), \( H^\infty(D) \), etc. (See [7].)

**Theorem 2.** Let \( X \) be a Grothendieck space with the Dunford-Pettis property, and let \( A, \{ A_\alpha \} \), and \( \{ B_\alpha \} \) be as in Theorem 1. If \( \{ A_\alpha \} \) is strongly ergodic, then it is uniformly ergodic.

Thus, in this case the conditions \((1)\)-(7) all are equivalent to each of \( D(P) = X \) and \( D(B) = R(A) \). In view of the next theorem, we have another equivalent condition: \( \overline{R(A^*)} = w^* - \text{cl}(R(A^*)) \).
Theorem 3. Let $X$ be a Grothendieck space, and let $A$, $\{A_\alpha\}$, and $\{B_\alpha\}$ be as in Theorem 1. Then $\{A_\alpha\}$ is strongly ergodic if and only if $\overline{R(A^*)} = w^* - \text{cl}(R(A^*))$.

2. Proof of Main Result

Proof of Theorem 1. We prove the implication $s: (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2)$, $(1) \Rightarrow (3) \Rightarrow (6) \Rightarrow (1) + (7)$.

(2) $\Rightarrow$ (3). (C3) implies that $\overline{R(A)}$ is invariant under $A_\alpha$ and $PA = 0$, so that $\|A_\alpha|\overline{R(A)}\| = \|(A_\alpha - P)|\overline{R(A)}\| \leq \|A_\alpha - P\| \to 0$. Hence for some $\alpha$, $(A_\alpha - I)|\overline{R(A)}$ is invertible and so $\overline{R(A)} \subset R(A_\alpha - I) \subset R(A)$.

(3) $\Rightarrow$ (4). By the open mapping theorem (cf. [15, p. 213]), there is some $m > 0$ such that each $x \in R(A)$ is equal to $Ay$ for some $y \in D(A)$ with $\|y\| \leq m\|x\|$. Hence $\|A_\alpha x\| = \|AA_\alpha y\| \leq \|A_\alpha\|m\|x\|$, showing that $\|A_\alpha|R(A)\| \leq m\|A_\alpha A_\alpha\| \to 0$ and so $(A_\alpha - I)|R(A)$ is invertible for some $\alpha$. This, together with (C2) and (C3), implies that $R(A) = (A_\alpha - I)AD(A) = A(A_\alpha - I)D(A) \subset D(D(A) \cap R(A)) = R(A^2)$. Hence $R(A^2) = R(A)$ and is closed.

(4) $\Rightarrow$ (5). If $x \in D(A)$, then $Ax = \lim\{A_\alpha x - A(A_\alpha - I)x\} = \lim (A_\alpha - I)x \in \overline{R(A^2)} = R(A^2)$. This shows that $R(A) = R(A^2)$ is closed and $D(A) \subset N(A) \cap R(A)$. Next, let $x \in X$, and let $\{x_n\}$ be a sequence in $D(A)$ such that $x_n \to x$. Then $A_\alpha x \in D(A)$ and $(A_\alpha - I)x = \lim_{n \to \infty} (A_\alpha - I)x_n \in \overline{R(A)} = R(A)$ so that $x = A_\alpha x - (A_\alpha - I)x \in D(A) + R(A) \subset N(A) + R(A)$. Hence $X = N(A) + R(A)$. To see that this is a direct sum, let $x \in N(A) \cap R(A)$. Then there is a $y$ such that $Ay = x \in N(A) \subset N(A_\alpha - I)$ for all $\alpha$. But then $x = A_\alpha x = A_\alpha Ay \to 0$.

(5) $\Rightarrow$ (2). The closedness of $A$ and assumption (5) imply $R(A)$ is closed (see [16, p. 217]). Then, as was shown in (3) $\Rightarrow$ (4), we have $\|(A_\alpha - P)|R(A)\| = \|A_\alpha|R(A)\| \leq m\|A_\alpha A_\alpha\| \to 0$. Because $(A_\alpha - P)|N(A) = 0$, it follows that $\|A_\alpha - P\| \to 0$.

(1) $\Rightarrow$ (3). Since $A_\alpha A \subset AA_\alpha$, the space $D(P) = N(A) \oplus \overline{R(A)}$ is invariant under $A_\alpha$, and $A(D(P))$ is a densely defined closed operator in $D(P)$. Applying (2) $\Rightarrow$ (5) to $\{A_\alpha|D(P)\}$ and $A|D(P)$, we infer that $D(P) = N(A|D(P)) \oplus R(A|D(P))$. Since $N(A|D(P)) = N(A)$ and $R(A|D(P)) \subset R(A)$, it follows from the two expressions of $D(P)$ that $\overline{R(A)} = R(A|D(P))$ and hence $\overline{R(A)} = R(A)$.

(3) $\Rightarrow$ (6). If $R(A)$ is closed, then as shown in (3) $\Rightarrow$ (4), we have $R(A^2) = \overline{R(A)}$, so that $D(B) = A(D(A) \cap R(A)) = R(A^2) = R(A)$ is closed. Since $B_\alpha y \to By$ for all $y \in D(B)$, the uniform boundedness principle implies (6).

(6)$\Rightarrow$ (1) + (7). (6) implies that $B$ is bounded and hence $R(A|D(P)) = A(D(A) \cap \overline{R(A)}) = D(B)$ is closed. An application of (3) $\Rightarrow$ (2) to $\{A_\alpha|D(P)\}$ asserts that $\|A_\alpha|D(P) - P\| \to 0$, from which, together with (7), we see that $\|B_\alpha|D(B) - B\| \leq \|A_\alpha|D(P) - P\| \|B\| \to 0$.

Finally, if (1)–(7) hold, then for any $x \in X$, we have $x - Px \in R(A) = R(A^2) = D(B)$ so that $AB(x - Px) = x - Px$ and

$$\|A_\alpha x - Px\| = \|A_\alpha(x - Px)\| = \|A_\alpha AB(I - P)x\| \leq \|AA_\alpha\|\|B\|(M + 1)\|x\|.$$ Hence $\|A_\alpha - P\| \leq (M + 1)\|AA_\alpha\|\|B\|$ and $\|B_\alpha|D(B) - B\| \leq (M + 1)\|AA_\alpha\|\|B\|^2$.

To prove Theorems 2 and 3 we need the following lemma.
Lemma. If $A$, \{${A}_\alpha$\}, and \{${B}_\alpha$\} satisfy conditions (C1), (C2), and (C3), then $A^*$, \{${A}^*_\alpha$\}, and \{${B}^*_\alpha$\} also satisfy these conditions.

Proof. It suffices to show that if $E \in B(X)$ is such that $R(E) \subset D(A)$ and $EA \subset AE$, then $R(E^*) \subset D(A^*)$ and $E^*A^* \subset A^*E^* = (AE)^*$. If $x^* \in D(A^*)$, then $\langle Ax^*, E^*x^* \rangle = \langle E Ax^*, x^* \rangle = \langle x^*, A^*E^*x^* \rangle$ for all $x \in D(A)$, so that $E^*x^* \in D(A^*)$ and $A^*E^*x^* = E^*A^*x^* = (AE)^*x^*$. Hence $E^*A^* \subset A^*E^*$. To show $R(E^*) \subset D(A^*)$ and $A^*E^* = (AE)^*$, we use the fact that $A^*$ is weakly densely defined and $w^* - w^*$-closed. For any $x^* \in X^*$, let $\{x^*_\alpha\}$ be a net in $D(A^*)$ such that $x^*_\alpha \to x^*$ weakly*. Then $E^*x^*_\alpha$ and $A^*E^*x^*_\alpha = (AE)^*x^*_\alpha$ converges weakly* to $E^*x^*$ and $(AE)^*x^*$, respectively. This implies that $E^*x^* \in D(A^*)$ and $A^*E^*x^* = (AE)^*x^*$.

Proof of Theorem 2. Since $A_\alpha |R(P) = I$, we may assume $A_\alpha \to 0$ strongly and show that $\|A_\alpha\| \to 0$, without loss of generality. Take a sequence $A_n \equiv A_\alpha \to 0$. Then $A_n x$ converges strongly to 0 for all $x \in X$. This implies that $\{A^*_n x^*_n\}$ converges weakly* and hence weakly to 0 whenever $\{x^*_n\}$ is bounded. In particular, $\{A^*_n x^*_n\} \to 0$ weakly for all $x^* \in X^*$. The convergence actually holds in the strong topology, by the strong ergodic theorem (applied to $\{A^*_n\}$). This fact in turn implies that $\{A_n x_n\}$ converges weakly to 0 whenever $\{x_n\}$ is bounded. Now, it follows from a lemma of Lotz [7] that $\|A^*_n\| \to 0$. Thus $I - A_m$ is invertible for a sufficiently large $m$. By (C2) and (C3) we obtain that

$$
\|A_n\| = \|A_n(I - A_m)(I - A_m)^{-1}\| = \|A_n AB_m(I - A_m)^{-1}\| \leq \|A_n\| \|B_m\| \|(I - A_m)^{-1}\| \to 0 \quad \text{as } n \to \infty.
$$

Application of Theorem 1 to $\{A_n\}$ and $\{A_\alpha\}$ shows first that $R(A)$ is closed and then that $\|A_\alpha\| \to 0$.

Proof of Theorem 3. If $D(P) = X$, then for every $x^* \in X^*$ we have

$$
w^* - \lim_{n \to \infty} A^*_n x^* = w^* - \lim_{n \to \infty} A^*_\alpha x^* = P^* x^*,
$$

where $\{A_\alpha\}$ is any subsequence of $\{A_\alpha\}$. The strong ergodic theorem applied to $\{A^*_\alpha\}$, shows that $\ast P^* x^* = s^* \lim_{n \to \infty} A^*_\alpha x^*$ for all $x^* \in X^*$, $N(P^*) = \overline{R(A^*)}$. Hence $\overline{R(A^*)} = R(P) = N(A) = [\overline{R(A^*)}]^\perp = w^* - \text{cl} (R(A^*))$.

Conversely, if $\overline{R(A^*)} = w^* - \text{cl} (R(A^*))$, then $D(P)^\perp = \{N(A) \oplus \overline{R(A^*)}\}^\perp = [\overline{R(A^*)}]^\perp = w^* - \text{cl} (R(A^*)) \cap N(A^*) = \overline{R(A^*)} \cap N(A^*) = \{0\}$, again following from the strong ergodic theorem applied to $\{A^*_\alpha\}$. Since $D(P)$ is closed, it must be equal to $X$.

3. Examples

We consider applications to $n$-times integrated semigroups, $(Y)$-semigroups, and cosine operator functions.

3.1. $n$-times integrated semigroups. Let $n$ be a positive integer. A strongly continuous family $\{T(t) ; t \geq 0\}$ in $B(X)$ is called an $n$-times integrated semigroup (see [1, 15]) if $T(0) = I$ and

$$
T(t)T(s) = \frac{1}{(n - 1)!} \left( \int_t^{t+s} (t + s - r)^{n-1} T(r) \, dr - \int_0^s (t + s - r)^{n-1} T(r) \, dr \right) \quad (s, t \geq 0).
$$
A $C_0$-semigroup is called an $o$-times integrated semigroup. It is known that the integrals over $[0, t], t \geq 0$, of an $n$-times ($n \geq 0$) integrated semigroup form an $(n + 1)$-times integrated semigroup, but not conversely.

$T(\cdot)$ is called nondegenerate if $T(t)x = 0$ for all $t > 0$ implies $x = 0$. It is called exponentially bounded if there are $M \geq 0$, $w \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{wt}$ for all $t \geq 0$. If $T(\cdot)$ is nondegenerate and exponentially bounded, then there exists a unique closed operator $A$ satisfying $(w, \infty) \subset \rho(A)$ and $(\lambda - A)^{-1}x = \int_0^\infty \lambda^n e^{-\lambda t}T(t)dt$ for $x \in X$ and $\lambda > w$. This operator is called the generator of $T(\cdot)$. It is not necessarily densely defined. We only consider the case when $A$ is densely defined. This includes all $C_0$-semigroups.

It is known [1, Proposition 3.3] that $\int_0^t T(s)xds \in D(A)$ and $\int_0^t T(s)xds = T(t)x = (t^n/n!)x$ for all $x \in X$, and $\int_0^t T(s)Axds = T(t)x - (t^n/n!)x$ for all $x \in D(A)$. Since $A$ is closed, taking integration gives that

$$
\int_0^t T(s)xds - (t^n/(n + 1)!)x = \begin{cases}
A\int_0^t \int_0^s T(u)duds & \text{for } x \in X \\
\int_0^t \int_0^s T(u)Axduds & \text{for } x \in D(A).
\end{cases}
$$

Let $A_t := (n + 1)!t^{-n-1}\int_0^t T(s)ds$ and $B_t := -(n + 1)!t^{-n-1}\int_0^t \int_0^s T(u)duds$ for $t > 0$. Then $B_tA \subset AB_t = I - A_t$ and $A_tA \subset AA_t = (n+1)!T(t)/t^{n+1}-(n+1)I/t$. Suppose $\|T(t)\| = O(t^n)$ ($t \to \infty$). Then $A$, $\{A_t\}$, and $\{B_t\}$ satisfy (C1), (C2), and (C3) as $t \to \infty$. On the other hand, the systems $\{\lambda(\lambda - A)^{-1}\}$, $\{-\lambda(\lambda - A)^{-1}\}$ clearly satisfy (C1), (C2), and (C3) as $\lambda \to 0$ too. Hence the strong ergodic theorem and the theorems in §1 are applicable to $\{A_t\}$ with $\{B_t\}$ and $\{\lambda(\lambda - A)^{-1}\}$ with $\{-\lambda(\lambda - A)^{-1}\}$, and the next two theorems follow immediately.

**Theorem 4.** Let $\{T(t); t \geq 0\}$ be a nondegenerate $n$-times integrated semigroup with generator $A$ densely defined. Suppose $\|T(t)\| = O(t^n)$ ($t \to \infty$). Let $A_t$ and $B_t$ be as previously defined. Then $s\text{-lim}_{t \to \infty} A_t x$ and $s\text{-lim}_{\lambda \to 0^+} \lambda(\lambda - A)^{-1}x$ exist and are equal if one of them exists, and the limits define a bounded linear projection $P$ onto $N(A)$ along $R(A)$. For $y \in R(A)$, $s\text{-lim}_{t \to \infty} B_t y$ and $s\text{-lim}_{\lambda \to 0^+} (A - \lambda)^{-1}y$ exist and are equal if one of them exists, and the limits define an operator $B$ which sends each $y \in A(D(A) \cap R(A))$ to the unique solution $x = By$ of $Ax = y$ in $R(A)$.

**Theorem 5.** Under the hypothesis of Theorem 4, the following statements are equivalent:

1. $\|A_t - P\| \to 0$ as $t \to \infty$,
2. $\|\lambda(\lambda - A)^{-1} - P\| \to 0$ as $t \to 0^+$,
3. $R(A)$ is closed,
4. $\|B_t|R(A)|\| = O(1)$ ($t \to \infty$),
5. $\|B_t|R(A) - B\| \to 0$ as $t \to \infty$,
6. $\|((\lambda - A)^{-1})R(A) - B\| \to 0$ as $\lambda \to 0^+$.

Moreover, when $X$ is a Grothendieck space with the Dunford-Pettis property, $D(P) = X$ and $R(A^*) = w^*\text{-cl}(R(A^*))$ are two more equivalent conditions.

**Remarks.** (i) When (1)-(6) hold, we have $\|A_t - P\| = O(1/t)$, $\|B_t|R(A) - B\| = O(1/t)$ ($t \to \infty$), and $\|\lambda(\lambda - A)^{-1} - P\| = O(\lambda)$, $\|((\lambda - A)^{-1})R(A) - B\| = O(\lambda)$, $\lambda \to 0^+$.
(ii) In the case \( n = 0 \), Theorem 4 is well known (see [3, pp. 58-60] for the first part, and [4] for the second part), the equivalence of (1), (2), and (3) in Theorem 5 is proved in [6] (see also [10]), the equivalence of strong ergodicity and \( R(A^*) = w^*\text{-cl}(RA^*) \) in a Grothendieck space is proved in [9], and the equivalence of strong ergodicity and uniform ergodicity in a Grothendieck space with the Dunford-Pettis property is proved in [7]. The theorems with \( n \geq 1 \) are new.

3.2. \((Y)\)-semigroups. Let \( Y \) be a closed subspace of \( X^* \) such that the canonical imbedding of \( X \) into \( Y^* \) is isometric. A semigroup \( \{T(t); t \geq 0\} \) of operators on \( X \) is called a \((Y)\)-semigroup (cf. [8, 11]) if \( Y \) is invariant under \( T^*(t) \) for all \( t \geq 0 \) and \( T^*(\cdot)x \) is \( \sigma(X, Y)\)-continuous on \( [0, \infty) \) and locally \( \sigma(X, Y)\)-Pettis integrable for all \( x \in X \). The generator \( A \) of \( T(\cdot) \) is defined by \( Ax := \sigma(X, Y)\)-lim_{\( t \to \infty \)} t^{-1}(T(t) - I)x \). A \( C_0 \)-semigroup on \( X \) is a \((X^*)\)-semigroup, and its dual semigroup is a \((X)\)-semigroup. The tensor product \( T(t) \) of two \( C_0 \)-semigroups \( e^{tA} \) and \( e^{-tB} \) on \( X \) is a \((Y)\)-semigroup on \( B(X) \) for some suitable subspace \( Y \) of \( B(X)^* \); its generator is the operator \( \Delta : C \to AC - CB \).

The strong convergence of ergodic limits of a \((Y)\)-semigroup and that of approximate solutions of the corresponding equation \( Ax = y \) have been discussed in [13, Example VI]. The result is the same as Theorem 4 with \( n = 0 \). By applying Theorems 1 and 2 one can easily see that Theorem 5 with \( n = 0 \) holds for \((Y)\)-semigroups too. Since \( S(t) := \int_0^t T(s) ds \), \( t \geq 0 \), forms a once-integrated semigroup, we can apply Theorems 4 and 5 to \( S(\cdot) \) to obtain ergodic theorems for \((C, 2)\)-means of \( T(\cdot) \); they are Theorems 4 and 5 with \( A_t = 2t^{-2} \int_0^t \int_0^s T(u) du ds \) and \( B_t = -2t^{-2} \int_0^t \int_0^s \int_0^u T(v) dv du ds \).

3.3. Cosine operator functions. A strongly continuous family \( \{C(t); t \in \mathbb{R}\} \) in \( B(X) \) is called a cosine operator function if \( C(0) = I \) and \( C(t+s) + C(t-s) = 2C(t)C(s) \), \( s, t \in \mathbb{R} \). The generator \( A \), defined by \( Ax := C''(0)x \), is a densely defined closed operator.

For \( t > 0 \) let
\[
A_t := 2t^{-2} \int_0^t \int_0^s C(u) du ds
\]
and
\[
B_t = -2t^{-2} \int_0^t \int_0^s \int_0^u C(w) dw du ds.
\]
Then we have \( B_t A \subset AB_t \subset I_t - A \) and \( A_t A \subset AA_t = 2t^{-2}(C(t) - I) \). The strong convergence of \( A_t x \) and \( B_t y \) as \( t \to \infty \) has been discussed in [13, Example VII]. We now deduce from Theorems 1 and 2 the following theorem about uniform convergence.

**Theorem 6.** Suppose that \( \| \int_0^t \int_0^s C(u) du ds \| = O(t^2) \quad (t \to \infty) \) and \( \| C(t) \| = o(t^2) \quad (t \to \infty) \). Then, with \( A_t \) and \( B_t \) defined as above, the conclusion of Theorem 5 remains valid.

**Concluding remark**

Our Theorems 1, 2, and 3 can also be used to deduce uniform ergodic theorems for discrete semigroups (cf. [5, 7]) and uniform ergodic theorems for pseudoresolvents (cf. [10, 12]).
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