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LUSTERNIK-SCHNIRELMANN CATEGORY OF RIBBON KNOT COMPLEMENT

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Dedicated to Professor Tokushi Nakamura on his sixtieth birthday

ABSTRACT. We showed in [8] that a locally flat knot is topologically unknotted if and only if the Lusternik-Schnirelmann category of the complement is one. In this paper we will show that the complement of a ribbon knot is of category two.

1. Review of known results

The category $\operatorname{cat} X$ of a space X is the least integer n so that X may be covered by the $n+1$ open subsets each of which is contractible to a point in X; if there is no such integer, $\operatorname{cat} X = \infty$. In particular, $\operatorname{cat} X = 0$ if and only if X is itself contractible. So, $\operatorname{cat} S^n = 1$. This definition is due to Fox [4], although we reduced the number by one from his definition. Originally Lusternik-Schnirelmann defined the category by the closed covering and these two notions coincide with each other for ANR's (cf. [5, §3]).

One of the most important properties is that $\operatorname{cat} X$ is a homotopy type invariant of X. Using this we can easily prove that $\operatorname{cat} X \leq \dim X$ for a connected CW complex X.

To estimate the category of a CW complex X from below we use the property (BGW) that if $\operatorname{cat} X \leq n$ then the cup product $u_0 \cup \cdots \cup u_n$ vanishes for $u_k \in H^*(X)$ $(k = 0, \ldots, n)$ with $* \geq 1$. We have immediately for example that $\operatorname{cat}(S^{n_1} \times \cdots \times S^{n_k}) = k$.

Moreover, a closed manifold M is a homotopy sphere if and only if $\operatorname{cat} M = 1$. In fact, if $\operatorname{cat} M = 1$ then M is a homology sphere by the property (BGW) and Poincaré duality theorem. So, it suffices to apply the lemma (cf. [4, §23]: If X is an arcwise connected and locally 1-connected compact metrizable space and $\operatorname{cat} X \leq 1$, then $\pi_1(X)$ is a free group.

A corresponding theorem in the case of knot complement is

Theorem 1 ([8]). *A locally flat knot $S^n \supset S^m$ is topologically unknotted if and only if* $\operatorname{cat}(S^n - S^m) = 1$.

By the facts known in geometric topology we have only to show that

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$\operatorname{cat}(S^n - S^{n-2}) = 1$ implies $S^n - S^{n-2} \simeq S^1$ and a detailed proof is given in [8].

2. Ribbon knot complement

Any nontrivial classical knot is an example of knot whose complement is of category two. More examples are given by simple fibered odd-dimensional knots. We will show that the ribbon knots are the other examples.

Definition. A knot obtained by making a band sum of a unlinked unknotted link of m components along $m-1$ mutually disjoint bands connecting the components is called a ribbon knot. Push the interior of the disjoint disks $D^{n-1} \cup \cdots \cup D^{n-1}$ bounded by the above trivial link into the interior of the disk D^{n+1}. A properly embedded disk pair $D^{n-1} \subset D^{n+1}$ obtained by taking a boundary band sum of the properly embedded disks $D^{n-1} \cup \cdots \cup D^{n-1}$ along the bands is called a ribbon disk pair. For a detailed definition we refer to [1].

Theorem 2. *For a nontrivial ribbon knot* $S^n \supset S^{n-2}$ *we have* $\operatorname{cat}(S^n - S^{n-2}) = 2$.

To prove the theorem we will use the following two observations about the exteriors. By the exterior we mean a compact manifold obtained by removing the interior of a regular neighborhood of a knot or a disk. Note that the complement and the exterior have the same homotopy type.

Observation 1 (Asano-Marumoto-Yanagawa [1, Proposition 2.6]). Let $RD = (D^{n-1} \subset D^{n+1})$ be a ribbon disk pair and $\partial RD = (\partial D^{n-1} \subset \partial D^{n+1})$ its boundary ribbon knot. Then,

$$\text{Exterior of } RD \bigcup_{\text{Exterior of } \partial RD} \text{Exterior of } RD = \text{Exterior of } RK$$

for a one higher dimensional ribbon knot $RK = (S^{n-1} \subset S^{n+1})$. And if $n \geq 3$ any ribbon knot $S^{n-1} \subset S^{n+1}$ is obtained as RK.

Observation 2 (Asano-Marumoto-Yanagawa [1, Proposition 2.3 and Theorem 2.4]). The exterior of RD is obtained as follows. Add $m+1$ 1-handles and canceling standard $m+1$ 2-handles $h_0^2 \cup \cdots \cup h_m^2$ to the original disk D_0^{n+1}. Then we can obtain m 2-handles $h_1'^2 \cup \cdots \cup h_m'^2$ by sliding the 2-handles other than h_0^2 over the 2-handle h_0^2 one after the other corresponding to the bands so that the exterior of RD is $D_0^{n+1} \cup h_0^1 \cup \cdots \cup h_m^1 \cup h_1'^2 \cup \cdots \cup h_m'^2$. If we add h_0^2 moreover then it reduces to D^{n+1} again and the dual axis $0 \times D^{n-1}$ of $D^2 \times D^{n-1}$ corresponding to this h_0^2 gives D^{n-1} of RD. In particular, the exterior of RD has the homotopy type of a 2-dimensional CW complex K^2.

Observations 1 and 2 together imply that the exterior of $RK = (S^{n-1} \subset S^{n+1})$ is obtained by adding $m+1$ 1-handles, m 2-handles, m $(n-1)$-(dual 2)-handles and m n-(dual 1)-handles to D_0^{n+1}. In fact, since each $(n-1)$-handle has no intersection with h_0^2, we can define W^k to be the manifold obtained by attaching up to k-handles for $k \leq n-1$. Note that K^2 in the observation 2 is the spine of W^2. Note also that we have a canonical identification $f : \partial W^{n-1} \to \partial W^1$ and that the 1-handles $h_1^1, \ldots, h_m^1$ have no intersection with $\partial_- h_0^2$. So, their dual n-handles do not touch $f^{-1}(\partial_- h_0^2)$ in ∂W^{n-1} and we get W^n. The intersection $\partial W^n \cap \text{dual } h_0^1 = f^{-1}(\partial W^1 \cap h_0^1 - \text{Int}(\partial_- h_0^2))$ is an n-disk and the intersection $\partial(W^n \cup \text{dual } h_0^1) \cap \text{dual } D_0^{n+1}$ is also an n-disk; we may think that the exterior of RK is W^n.

Note here that the axis D^{n-1} of an $(n-1)$-handle attaches at the boundary of dual axis of the corresponding 2-handle so as to make double. So, adding each $(n-1)$-handle changes the homotopy type of K^2 by one point attaching of S^{n-1}. Hence,

$$W^{n-1} \simeq K^{n-1} = K^2 \vee S^{n-1} \vee \cdots \vee S^{n-1} \qquad (n \geq 3).$$

Each n-handle is attached along an element α of $\pi_{n-1}(K^2 \vee S^{n-1} \vee \cdots \vee S^{n-1})$ to get $W^n \simeq K^n = K^{n-1} \cup (e^n \cup \cdots \cup e^n)$. Denoting the 1-skeleton of K^2 by $S^1 \vee \cdots \vee S^1$ it is not difficult to see that the natural map

$$\begin{aligned} i_* + j_* : \pi_{n-1}(K^2) \oplus \pi_{n-1}(S^1 \vee \cdots \vee S^1 \vee S^{n-1} \vee \cdots \vee S^{n-1}) \\ \longrightarrow \pi_{n-1}(K^2 \vee S^{n-1} \vee \cdots \vee S^{n-1}) \end{aligned}$$

induced by the inclusions is a surjection. We will prove that α belongs to the image of j_*. If $\pi_{n-1}(K^2) = 0$ as the Whitehead conjecture says, we have nothing to prove. Now note that collapsing each of m number of S^{n-1} in $K^{n-1} \subset K^n$ into a point is realized by adding no $(n-1)$-handles and attaching the axis of each n-handle at the boundary of dual axis of the corresponding 1-handle in the course of constructing W^n. Let c denote the restriction of the collapsing map on K^{n-1}. Then, the image of the attaching map α by the induced map $c_* : \pi_{n-1}(K^{n-1}) \to \pi_{n-1}(K^2)$ is zero, because the dual axis is contractible. Since the composition of the restriction i_* of $i_* + j_*$ on $\pi_{n-1}(K^2)$ with c_* is the identity map, we see that α is contained in the image of the restriction j_* of $i_* + j_*$ on $\pi_{n-1}(S^1 \vee \cdots \vee S^1 \vee S^{n-1} \vee \cdots \vee S^{n-1})$. Hence, we have

$$W^n \simeq K' = (S^1 \vee \cdots \vee S^1 \vee S^{n-1} \vee \cdots \vee S^{n-1}) \cup (e^2 \cup \cdots \cup e^2 \cup e^n \cup \cdots \cup e^n).$$

A standard neighborhood of $S^1 \vee \cdots \vee S^1 \vee S^{n-1} \vee \cdots \vee S^{n-1}$ in K' is of category one and $\text{Int}(e^2 \cup \cdots \cup e^2 \cup e^n \cup \cdots \cup e^n)$ is contractible to a point in K'. Therefore, $\text{cat}\, W^n \leq 2$, that is, the category of the complement of a ribbon knot $RK = (S^{n+1} \supset S^{n-1})$ is less than or equal to two, provided that $n \geq 3$. Since Theorem 2 is true for any classical knot, this completes a proof of Theorem 2 due to Theorem 1.

3. Comments

We have examples of closed connected n-dimensional manifolds with any integer from 1 to n as their category; $\text{cat}(S^{n-k+1} \times T^{k-1}) = k$. I conjecture that there would be also an example of knot $S^{n-2} \subset S^n$ whose complement has a given category from 3 to $n-1$. Actually the complement of a fibered knot with fiber a punctured torus $T^{n-1} - pt$, which is known to exist for $n = 4, 5, 6$ [3], is of category $n-1$ or possibly $n-2$. The complement of its spun knot may be expected to have the same category.

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