LUSTERNIK-SCHNIRELMANN CATEGORY OF RIBBON KNOT COMPLEMENT

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Dedicated to Professor Tokushi Nakamura on his sixtieth birthday

Abstract. We showed in [8] that a locally flat knot is topologically unknotted if and only if the Lusternik-Schnirelmann category of the complement is one. In this paper we will show that the complement of a ribbon knot is of category two.

1. Review of known results

The category $\text{cat } X$ of a space $X$ is the least integer $n$ so that $X$ may be covered by the $n + 1$ open subsets each of which is contractible to a point in $X$; if there is no such integer, $\text{cat } X = \infty$. In particular, $\text{cat } X = 0$ if and only if $X$ is itself contractible. So, $\text{cat } S^n = 1$. This definition is due to Fox [4], although we reduced the number by one from his definition. Originally Lusternik-Schnirelmann defined the category by the closed covering and these two notions coincide with each other for ANR's (cf. [5, §3]).

One of the most important properties is that $\text{cat } X$ is a homotopy type invariant of $X$. Using this we can easily prove that $\text{cat } X \leq \dim X$ for a connected CW complex $X$.

To estimate the category of a CW complex $X$ from below we use the property (BGW) that if $\text{cat } X \leq n$ then the cup product $u_0 \cup \cdots \cup u_n$ vanishes for $u_k \in H^*(X)$ ($k = 0, \ldots, n$) with $* \geq 1$. We have immediately for example that $\text{cat } (S^{n_1} \times \cdots \times S^{n_k}) = k$.

Moreover, a closed manifold $M$ is a homotopy sphere if and only if $\text{cat } M = 1$. In fact, if $\text{cat } M = 1$ then $M$ is a homology sphere by the property (BGW) and Poincaré duality theorem. So, it suffices to apply the lemma (cf. [4, §23]): If $X$ is an arcwise connected and locally 1-connected compact metrizable space and $\text{cat } X \leq 1$, then $\pi_1(X)$ is a free group.

A corresponding theorem in the case of knot complement is

Theorem 1 ([8]). A locally flat knot $S^n \supset S^m$ is topologically unknotted if and only if $\text{cat } (S^n - S^m) = 1$.

By the facts known in geometric topology we have only to show that...
cat($S^n - S^{n-2}$) = 1 implies $S^n - S^{n-2} \simeq S^1$ and a detailed proof is given in [8].

2. RIBBON KNOT COMPLEMENT

Any nontrivial classical knot is an example of knot whose complement is of category two. More examples are given by simple fibered odd-dimensional knots. We will show that the ribbon knots are the other examples.

**Definition.** A knot obtained by making a band sum of a unlinked unknotted link of $m$ components along $m - 1$ mutually disjoint bands connecting the components is called a ribbon knot. Push the interior of the disjoint disks $D^{n-1} \cup \cdots \cup D^{n-1}$ bounded by the above trivial link into the interior of the disk $D^{n+1}$. A properly embedded disk pair $D^{n-1} \subset D^{n+1}$ obtained by taking a boundary band sum of the properly embedded disks $D^{n-1} \cup \cdots \cup D^{n-1}$ along the bands is called a ribbon disk pair. For a detailed definition we refer to [1].

**Theorem 2.** For a nontrivial ribbon knot $S^n \supset S^{n-2}$ we have $\text{cat}(S^n - S^{n-2}) = 2$.

To prove the theorem we will use the following two observations about the exteriors. By the exterior we mean a compact manifold obtained by removing the interior of a regular neighborhood of a knot or a disk. Note that the complement and the exterior have the same homotopy type.

**Observation 1** (Asano-Marumoto-Yanagawa [1, Proposition 2.6]). Let $RD = (D^{n-1} \subset D^{n+1})$ be a ribbon disk pair and $\partial RD = (\partial D^{n-1} \subset \partial D^{n+1})$ its boundary ribbon knot. Then,

$$\text{Exterior of } RD \bigcup \text{Exterior of } \partial RD = \text{Exterior of } RD - \text{Exterior of } RK$$

for a one higher dimensional ribbon knot $RK = (S^{n-1} \subset S^{n+1})$. And if $n \geq 3$ any ribbon knot $S^{n-1} \subset S^{n+1}$ is obtained as $RK$.

**Observation 2** (Asano-Marumoto-Yanagawa [1, Proposition 2.3 and Theorem 2.4]). The exterior of $RD$ is obtained as follows. Add $m + 1$ 1-handles and canceling standard $m + 1$ 2-handles $h_0^2 \cup \cdots \cup h_m^2$ to the original disk $D_0^{n+1}$. Then we can obtain $m$ 2-handles $h_1^2 \cup \cdots \cup h_m^2$ by sliding the 2-handles other than $h_0^2$ over the 2-handle $h_0^2$ one after the other corresponding to the bands so that the exterior of $RD$ is $D_0^{n+1} \cup h_1^1 \cup \cdots \cup h_m^1 \cup h_1^2 \cup \cdots \cup h_m^2$. If we add $h_0^2$ moreover then it reduces to $D^{n+1}$ again and the dual axis $0 \times D^{n-1}$ of $D^2 \times D^{n-1}$ corresponding to this $h_0^2$ gives $D^{n-1}$ of $RD$. In particular, the exterior of $RD$ has the homotopy type of a 2-dimensional CW complex $K^2$.

Observations 1 and 2 together imply that the exterior of $RK = (S^{n-1} \subset S^{n+1})$ is obtained by adding $m + 1$ 1-handles, $m$ 2-handles, $m$ $(n - 1)$-(dual 2)-handles and $m$ $n$-(dual 1)-handles to $D_0^{n+1}$. In fact, since each $(n - 1)$-handle has no intersection with $h_0^2$, we can define $W^k$ to be the manifold obtained by attaching up to $k$-handles for $k \leq n - 1$. Note that $K^2$ in the observation 2 is the spine of $W^2$. Note also that we have a canonical identification $f : \partial W^{n-1} \rightarrow \partial W^1$ and that the 1-handles $h_1^1, \ldots, h_m^1$ have no intersection with $\partial h_0^2$. So, their dual $n$-handles do not touch $f^{-1}(\partial h_0^2)$ in $\partial W^{n-1}$ and we
get $W^n$. The intersection $\partial W^n \cap \text{dual } h_0^1 = f^{-1}(\partial W^1 \cap h_0^1 - \text{Int}(\partial h_0^3))$ is an $n$-disk and the intersection $\partial (W^n \cup \text{dual } h_0^1) \cap \text{dual } D_0^{n+1}$ is also an $n$-disk; we may think that the exterior of RK is $W^n$.

Note here that the axis $D_1$ of an $(n - 1)$-handle attaches at the boundary of dual axis of the corresponding 2-handle so as to make double. So, adding each $(n - 1)$-handle changes the homotopy type of $K^2$ by one point attaching of $S^{n-1}$. Hence,

$$W^{n-1} \simeq K^{n-1} = K^2 \vee S^{n-1} \vee \ldots \vee S^{n-1} \quad (n \geq 3).$$

Each $n$-handle is attached along an element $\alpha$ of $\pi_{n-1}(K^2 \vee S^{n-1} \vee \ldots \vee S^{n-1})$ to get $W^n \simeq K^n = K^{n-1} \cup (e^n \cup \ldots \cup e^n)$. Denoting the 1-skeleton of $K^2$ by $S^1 \vee \ldots \vee S^1$ it is not difficult to see that the natural map

$$i_* + j_* : \pi_{n-1}(K^2) \oplus \pi_{n-1}(S^1 \vee \ldots \vee S^1) \to \pi_{n-1}(K^2 \vee S^{n-1} \vee \ldots \vee S^{n-1})$$

induced by the inclusions is a surjection. We will prove that $\alpha$ belongs to the image of $j_*$. If $\pi_{n-1}(K^2) = 0$ as the Whitehead conjecture says, we have nothing to prove. Now note that collapsing each of $m$ number of $S^{n-1}$ in $K^{n-1} \subset K^n$ into a point is realized by adding no $(n - 1)$-handles and attaching the axis of each $n$-handle at the boundary of dual axis of the corresponding 1-handle in the course of constructing $W^n$. Let $c$ denote the restriction of the collapsing map on $K^{n-1}$. Then, the image of the attaching map $\alpha$ by the induced map $c_* : \pi_{n-1}(K^{n-1}) \to \pi_{n-1}(K^2)$ is zero, because the dual axis is contractible. Since the composition of the restriction $i_*$ of $i_* + j_*$ on $\pi_{n-1}(S^1 \vee \ldots \vee S^1)$ with $c_*$ is the identity map, we see that $\alpha$ is contained in the image of the restriction $j_*$ of $i_* + j_*$ on $\pi_{n-1}(S^1 \vee \ldots \vee S^1) \vee \ldots \vee S^{n-1})$. Hence, we have

$$W^n \simeq K' = (S^1 \vee \ldots \vee S^1 \vee S^{n-1} \vee \ldots \vee S^{n-1}) \cup (e^2 \cup \ldots \cup e^n \cup \ldots \cup e^n).$$

A standard neighborhood of $S^1 \vee \ldots \vee S^1 \vee S^{n-1} \vee \ldots \vee S^{n-1}$ in $K'$ is of category one and $\text{Int}(e^2 \cup \ldots \cup e^n \cup \ldots \cup e^n)$ is contractible to a point in $K'$. Therefore, $\text{cat } W^n \leq 2$, that is, the category of the complement of a ribbon knot $\text{RW} = (S^{n+1} \cup S^{n+1})$ is less than or equal to two, provided that $n \geq 3$. Since Theorem 2 is true for any classical knot, this completes a proof of Theorem 2 due to Theorem 1.

3. Comments

We have examples of closed connected $n$-dimensional manifolds with any integer from 1 to $n$ as their category; $\text{cat}(S^{n-k+1} \times T^{k-1}) = k$. I conjecture that there would be also an example of knot $S^{n-2} \subset S^n$ whose complement has a given category from 3 to $n - 1$. Actually the complement of a fibered knot with fiber a punctured torus $T^{n-1} - pt$, which is known to exist for $n = 4, 5, 6$ [3], is of category $n - 1$ or possibly $n - 2$. The complement of its spun knot may be expected to have the same category.
REFERENCES


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