CLOSURE SETS OF FUNCTIONS
AND A HIERARCHY OF FILTERS

ROBERT MIGNONE

(Communicated by Andreas R. Blass)

ABSTRACT. This paper presents a transfinite extension of a filter generating closure property of functions. One consequence of this extension is a hierarchy of filters which coincide with filters generated by a directed set type closure property. At each level of this hierarchy a progressively stronger notion of normality is satisfied. Ideals with this stronger notion of normality each have a corresponding saturation characterization.

Normality for filters on $P_{\kappa}\lambda$ can be defined in terms of regressive functions from $P_{\kappa}\lambda$ into $\lambda$. It is well known that an equivalent definition for normality can be given in terms of regressive functions from $P_{\kappa}\lambda$ into $P_{\omega}\lambda$. Likewise, the minimal normal filter can be characterized as the filter generated by sets closed under functions from $\lambda \times \lambda$ into $P_{\kappa}\lambda$ and an equivalent characterization can be made using functions from $P_{\omega}\lambda$ into $P_{\kappa}\lambda$. What if these two notions were extended to functions from $P_{\kappa}\lambda$ into $P_{\delta}\lambda$ and functions from $P_{\delta}\lambda$ into $P_{\delta}\lambda$ respectively, for appropriate $\delta < \kappa$? This question leads to the concepts and results in the present paper.

Normality for filters on $P_{\kappa}\lambda$ can be defined in terms of regressive functions from $P_{\kappa}\lambda$ into $\lambda$. It is well known that an equivalent definition for normality can be given in terms of regressive functions from $P_{\kappa}\lambda$ into $P_{\omega}\lambda$. Likewise, the minimal normal filter can be characterized as the filter generated by sets closed under functions from $\lambda \times \lambda$ into $P_{\kappa}\lambda$ and an equivalent characterization can be made using functions from $P_{\omega}\lambda$ into $P_{\kappa}\lambda$. What if these two notions were extended to functions from $P_{\kappa}\lambda$ into $P_{\delta}\lambda$ and functions from $P_{\delta}\lambda$ into $P_{\delta}\lambda$ respectively, for appropriate $\delta < \kappa$? This question leads to the concepts and results in the present paper.

Standard notation is used throughout. Let $\kappa$ be a regular cardinal. If $\lambda \geq \kappa$ is a cardinal with $cf(\lambda) \geq \kappa$,

$$P_{\kappa}\lambda = \{ x \subseteq \lambda : |x| < \kappa \}.$$ Given $x \in P_{\kappa}\lambda$, let $\hat{x} = \{ y \in P_{\kappa}\lambda : x \subseteq y \}$. If $F$ is a filter on $P_{\kappa}\lambda$, $F^*$ is its dual ideal and vice versa, and

$$F^* = \{ A \subseteq P_{\kappa}\lambda : A \notin F^* \}.$$ The final section filter is given by

$$FSFK_{\kappa}\lambda = \{ A \subseteq P_{\kappa}\lambda : \hat{x} \subseteq A \text{ for some } x \in P_{\kappa}\lambda \}.$$ Filters over $P_{\kappa}\lambda$ extending $FSFK_{\kappa}\lambda$ are called fine. And $A \subseteq P_{\kappa}\lambda$ is said to be unbounded if $A \cap \hat{x} \neq \emptyset$ for all $x \in P_{\kappa}\lambda$. For the extent of this paper, all filters

Received by the editors April 20, 1990 and, in revised form, September 14, 1990; the results contained in this paper were presented in University Park, Pennsylvania, April 1990, during the spring meeting of the American Mathematical Society.

1980 Mathematics Subject Classification (1985 Revision). Primary 04A20.

This research was partially supported by a College of Charleston Research Grant and a Southeastern Regional Education Board Grant. The author also wishes to express his appreciation to MSRI, where this research was carried out.

©1992 American Mathematical Society
0002-9939/92 $1.00 + .25$ per page

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
(ideals) will be \( \kappa \)-complete extensions of \( \mathcal{FSF}_{\kappa \lambda} \) ( \( \mathcal{FSF}_{\kappa \lambda}^\ast \)). A set \( C \subseteq P_\kappa \lambda \) is said to be \textit{closed under chains}, if whenever \( \{ x_\gamma : \gamma < \delta \} \subseteq C \) where \( x_\gamma \subseteq x_\gamma' \), when \( \gamma < \gamma' < \delta < \kappa \), then \( \bigcup \{ x_\gamma : \gamma < \delta \} \in C \). The \textit{closed unbounded filter} is given by

\[
\mathcal{CF}_{\kappa \lambda} = \{ A \subseteq P_\kappa \lambda : C \subseteq A \text{ for some closed unbounded set } C \}.
\]

Let

\[
\delta = \inf \{ \delta \leq \kappa : \exists \gamma < \kappa \text{ and } \gamma < \delta \geq \kappa \}.
\]

**Definition 1.** For \( \delta \) regular, let \( v : P_\delta \lambda \rightarrow P_\kappa \lambda \) and

\[
C(v) = \{ x \in P_\kappa \lambda : y \in P_\delta x \implies v(y) \in x \}.
\]

If \( x \in C(v) \), then \( x \) is said to be \textit{closed under} \( v \) and \( C(v) \) is called the \textit{closure set} of \( v \).

**Definition 2.** Let \( \delta \) be regular and \( \delta < \delta \). Given \( v : P_\delta \lambda \rightarrow P_\kappa \lambda \), define \( v' \) as follows: for \( x \in P_\kappa \lambda \), let

\[
v_1(x) = \bigcup \{ v(y) : y \in P_\delta x \} \cup x;
\]

and for \( 1 < \gamma < \delta \),

\[
v_{\gamma}(x) = \bigcup \left\{ v(y) : y \in P_\delta \left( \bigcup \{ v_\xi(x) : \xi < \gamma \} \right) \right\}.
\]

Set

\[
v'(x) = \bigcup_{\gamma < \delta} v_{\gamma}(x).
\]

**Definition 3.** For a regular \( \delta < \kappa \), \( D \subseteq P_\kappa \lambda \) is said to be \( \delta \)-\textit{directed}, if whenever \( d \subseteq D \) and \( |d| < \delta \), there exists a \( y \in D \) such that \( \bigcup d \subseteq y \). And \( C \subseteq P_\kappa \lambda \) is said to be \( \delta \)-\textit{d-closed}, if whenever \( D \subseteq C \) such that \( |D| < \kappa \) and \( D \) is \( \delta \)-directed, then \( \bigcup D \in C \).

**Theorem 4.** Let \( \delta < \delta \) be regular. Suppose that \( v : P_\delta \lambda \rightarrow P_\kappa \lambda \) and \( w : P_\delta \lambda \rightarrow P_\kappa \lambda \). Then

1. \( C(v) = C(v') \);
2. \( \forall x \in P_\kappa \lambda (x \subseteq v'(x) \land v'(x) \in C(v')) \);
3. \( \forall x \in P_\delta \lambda (v'(x) = \bigcup \{ v'(y) : y \in P_\delta (v'(x)) \}) \);
4. \( \forall x, y \in P_\delta \lambda ((v'(x) \cap v'(y)) \in C(v')) \);
5. \( \forall x \in P_\delta \lambda (C(w) \subseteq C(v) \iff v'(x) \subseteq w'(x)) \);
6. \( \forall x \in P_\delta \lambda (v'(x) \text{ is the smallest set closed under } v, \text{ containing } x) \);
7. \( C(v') = \{ \bigcup v'[x] : x \in P_\kappa \lambda \}, \text{ where } v'[x] = \{ v'(y) : y \in P_\delta x \} \);
8. \( v'[x] \text{ is } \delta \text{-directed} \).

**Proof.**

1. \( C(v') \subseteq C(v) \text{ since } \forall x \in P_\delta \lambda (v(x) \subseteq v'(x)) \). Next, given \( x \in C(v) \), let \( y \in P_\delta x \). Then by induction on \( \gamma < \delta \),

\[
v_1(y) = \bigcup \{ v(z) : z \in P_\delta y \} \cup y \subseteq x;
\]

and

\[
v_{\gamma}(y) = \bigcup \{ v(z) : z \in P_\delta (\bigcup \{ v_{\gamma'}(y) : \gamma' < \gamma \}) \} \subseteq x,
\]

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
since \( \forall y' < \gamma(v(y') \subset x) \), by the induction hypothesis. Hence, \( v'(y) \subset x \) and \( x \in C(v') \).

(2) Clearly, \( x \in P_\delta(v'(x)) \). And if \( y \in P_\delta(v'(x)) \), then \( |y| < \delta \), so \( y \subset v_\gamma(x) \) for some \( \gamma < \delta \). Hence \( v(y) \in v_{\gamma+1}(x) \).

(3) Let \( \alpha \in v'(x) \). Then \( \alpha \in v'({\alpha}) \subset \bigcup\{ v'(y) : y \in P_\delta(v'(x)) \} \). And if \( \alpha \in \bigcup\{ v'(y) : y \in P_\delta(v'(x)) \} \), then \( \alpha \in v'(y) \subset v'(x) \) by (2).

(4) This holds since \( v'(x) \) and \( v'(y) \) are in \( C(v) \) by (2).

(5) For the first direction, \( x \in P_\delta(w'(x)) \) and \( w'(x) \in C(v) = C(v') \) by (1). Hence, \( v'(x) \subset w'(x) \). For the remaining direction, if \( x \in C(w) \) and \( y \in P_\delta x \), then \( v'(y) \subset w'(y) \subset x \). Hence, \( x \in C(v) \).

(6) Given \( x \in P_\beta \), let \( z \in C(v) \) such that \( x \subset z \). Then \( v'(x) \subset z \).

(7) Let \( x \in C(v') \). Then \( x = \bigcup v'[x] \). This is so, since given \( \alpha \in x \), then \( \alpha \in v'({\alpha}) \in \{ v'(y) : y \in P_\delta x \} \). And if \( \alpha \in \bigcup v'[x] \), then \( \alpha \in v'(y) \) and \( y \in P_\delta x \), so \( \alpha \in x \). Next, let \( y \in P_\delta(\bigcup v'[x]) \). Then \( y \subset \bigcup\{ v'(z) : z \in P_\delta x \} \). Let \( z_\alpha \in P_\delta x \) such that \( \alpha \in v'(z_\alpha) \) for \( \alpha \in y \). So \( y \subset \bigcup\{ v'(z_\alpha) : \alpha \in y \} \subset v'(\bigcup z_\alpha : \alpha \in y) \). But \( v'(\bigcup z_\alpha : \alpha \in y) \subset C(v'); \) hence \( v'(y) \subset \bigcup v'[x] \).

(8) Let \( d = \{ v'(y') : y < \xi \} \subset v'[x] \), where \( y' \in P_\delta x \) and \( \xi < \delta \). Since \( \delta \) is regular, \( |\bigcup\{ y : y < \xi \}| < \delta \), and hence \( \bigcup\{ y : y < \xi \} \subset P_\delta x \). By (3), \( \bigcup d \subset v'(\bigcup\{ y : y < \xi \}) \). □

The next definition strengthens the notion of normality for filters on \( P_\kappa \).

**Definition 5.** A filter \( F \) on \( P_\kappa \) is \( \delta \)-normal, if whenever \( f : P_\kappa \rightarrow P_\kappa \) such that \( A \in F^+ \) and \( \forall x \in A(f(x) \in P_\delta x) \), then \( \exists B \subseteq A \), and \( y \in P_\delta \) such that \( B \in f^+ \) and \( f(x) = y \ \forall x \in B \).

As in the case for normality, \( \delta \)-normality can be characterized by a type of diagonal intersection. The symbol \( \Delta^\delta \) will be used to indicate the diagonal intersection of a collection of subsets of \( P_\kappa \) indexed over \( P_\delta \).

**Theorem 6.** A filter \( F \) on \( P_\kappa \) is \( \delta \)-normal if and only if whenever \( \{ C_y : y \in P_\delta \} \subset F \) then

\[
\Delta^\delta \{ C_y : y \in P_\delta \} = \{ x \in P_\kappa : y \in P_\delta x \implies x \in C_y \} \in F.
\]

**Proof.** The usual argument adapts to this situation, see [Je1]. □

In [Me], Menas showed that the filter generated by closure sets of functions \( v : P_\omega \lambda \rightarrow P_\kappa \lambda \) is the same as the closed unbounded filter, \( CF_{\kappa,\lambda} \), on \( P_\kappa \lambda \). In the same paper, Menas mentions and uses the fact that the filter generated by closure sets of functions \( v : P_\omega \lambda \rightarrow P_\kappa \lambda \) is the same as the filter generated by closure sets of functions \( v : \lambda \times \lambda \rightarrow P_\kappa \lambda \). This fact has proved useful in obtaining results involving the closed unbounded filter on \( P_\kappa \lambda \). In particular, Carr used it to prove that \( CF_{\kappa,\lambda} \) is the minimal normal filter on \( P_\kappa \lambda \), see [Ca]; Johnson used it to get results on \( M \)-ideals on \( P_\kappa \lambda \), see [Jo]. Also in [Me], is the slightly hidden fact that the filter generated by the closure sets of functions \( v : \lambda \rightarrow P_\kappa \lambda \) is the same as \( SCF_{\kappa,\lambda} \), the strongly closed unbounded filter (which is the filter generated by unbounded subsets of \( P_\kappa \lambda \) closed under arbitrary unions of size less than \( \kappa \) ). In [Mi], this fact and a theorem similar to Theorem 4 for \( v : \lambda \rightarrow P_\kappa \lambda \) are used to establish results about quasinormal filters on \( P_\kappa \lambda \).

In what follows, a general theory is developed for filters generated by \( v : P_\delta \lambda \rightarrow P_\kappa \lambda \), where \( \delta < \delta \) is regular. What emerges is a strict linear hierarchy
of filters (ideals), each coinciding with a corresponding \( \delta \)-closed unbounded filter and each satisfying a progressively stronger normality property, namely \( \delta \)-normality.

**Definition 7.** Let \( \delta < \delta \) be regular. The closure set filter is given by

\[
F^\delta_{k\lambda} = \{ A \subseteq P_{k\lambda} : \exists v : P_\delta \lambda \rightarrow P_{k\lambda} \text{ and } C(v) \subseteq A \}.
\]

**Theorem 8.** \( F^\delta_{k\lambda} \) is a \( \delta \)-normal, \( \kappa \)-complete fine filter on \( P_{k\lambda} \).

**Proof.** The verification that \( F^\delta_{k\lambda} \) is a \( \kappa \)-complete, fine filter is routine. Let

\[
C(v_y) = \{ A \subseteq P_{\delta k} : \exists \alpha \in A \text{ where } \alpha \in C(v) \}.
\]

Define \( v(z) = \bigcup \{ v'_y(z) : y \in P_\delta z \} \). Let \( x \in C(v) \), \( y \in P_\delta x \) and \( z \in P_\delta x \).

Now

\[
v(z \cup y) = \bigcup \{ v'_y(z \cup y) : t \in P_\delta (z \cup y) \}.
\]

This gives \( v_y(z) \subseteq v'_y(z \cup y) \subseteq v(z \cup y) \subseteq x \). The first inclusion follows from the construction of \( v'_y \) from \( v_y \), the second inclusion from the definition of \( v \) and the third from the fact that \( x \in C(v) \). \( \Box \)

**Definition 9.** Let \( \delta < \delta \), be regular.

\[
DCF^\delta_{k\lambda} = \{ A \subseteq P_{k\lambda} : \exists C \subseteq A \text{ where } C \text{ is } \delta \text{-dclosed unbounded} \}.
\]

**Note.** If \( C \subseteq P_{k\lambda} \) is \( \delta \)-dclosed unbounded, then \( C \) is closed under unions of chains from \( C \) of cofinal length greater than or equal to \( \delta \) (call this \( \delta \)-closed). The converse does not appear to yield to Solovay's method, see [Mag]. It would be interesting to determine whether or not a set is \( \delta \)-closed unbounded if and only if it is \( \delta \)-dclosed unbounded. Or, if the filter generated by \( \delta \)-dclosed unbounded sets is denoted by \( CF^\delta_{k\lambda} \), then \( DCF^\delta_{k\lambda} \subseteq CF^\delta_{k\lambda} \), but does equality hold?

Both \( DCF^\delta_{k\lambda} \) and \( CF^\delta_{k\lambda} \) are \( \kappa \)-complete, fine, \( \delta \)-normal filters on \( P_{k\lambda} \). For \( DCF^\delta_{k\lambda} \) it is a consequence of the next theorem and for \( CF^\delta_{k\lambda} \) the verification is routine and will be omitted.

**Theorem 10.** For \( \delta < \delta \), regular, \( F^\delta_{k\lambda} = DCF^\delta_{k\lambda} \).

**Proof.** Given \( v : P_\delta \lambda \rightarrow P_{k\lambda} \), let \( D \subseteq C(v) \) be \( \delta \)-directed, where \( |D| < \kappa \). Let \( x \in P_\delta (\bigcup D) \). For each \( \alpha \in x \), there exists a \( x_\alpha \in D \) such that \( \alpha \in x_\alpha \). There exists a \( y \in D \) such that \( \bigcup_{ \alpha \in x} x_\alpha \subseteq y \). So \( x \in P_\delta y \). This gives \( v(x) \subseteq y \subseteq \bigcup D \). Note, \( C(v) \) is unbounded by parts (1) and (2) of Theorem 4. Hence \( F^\delta_{k\lambda} \subseteq DCF^\delta_{k\lambda} \).

Next, assume \( A \subseteq P_{k\lambda} \) is closed under \( \delta \)-directed subsets of cardinality less than \( \kappa \), and unbounded. Define \( v_\gamma : P_\delta \lambda \rightarrow P_{k\lambda} \), for \( \gamma < \delta \), as follows: for each \( y \in P_\delta \lambda \),

\[
v_1(y) \in A, \quad \text{where } y \subseteq v_1(y);
\]

\[
v_\gamma(y) \in A, \quad \text{where } \bigcup \{ v_\gamma'(z) : \gamma' < \gamma \text{ and } z \in y \} \subseteq v_\gamma(y).
\]

**Claim.** For every \( x \in P_{k\lambda} \), \( D_x = \{ v_\gamma(y) : \gamma < \delta \text{ and } y \in P_\delta x \} \) is a \( \delta \)-directed subset of \( P_{k\lambda} \), of cardinality less than \( \kappa \).
Proof of claim. Since \( \delta < \delta' \), \( |D_x| < \kappa \). So let
\[
\{ v_\alpha(y) : \alpha < \delta' \} \subset D_x, \quad \text{where } \delta' < \delta.
\]
Setting \( \gamma = \sup\{ y_\alpha : \alpha < \delta' \} + 1 \) and \( y = \bigcup_{\alpha < \delta'} y_\alpha \), gives \( \gamma < \delta \) and \( y \in P_\delta x \).
By construction
\[
\bigcup\{ v_\alpha(y) : \alpha < \delta' \} \subset v_\gamma(y) \in D_x.
\]
This proves the claim.

Define \( v : P_\delta \lambda \rightarrow P_\kappa \lambda \) by \( v(y) = \bigcup D_y \). Let \( x \in C(v) \). The construction of \( v_\gamma \) and \( D_x \) yields \( x \subset \bigcup D_x \). And if \( \alpha \in \bigcup D_x \), then \( \alpha \in v_\gamma(y) \) for some \( y \in P_\delta x \). But
\[
v_\gamma(y) \subset \bigcup D_y = v(y) \subset x.
\]
Hence, \( \bigcup D_x \subset x \). Since \( A \) is closed under \( \delta \)-directed sets of cardinality less than \( \kappa \), by the claim, \( \bigcup D_x \in A \). Hence \( x \in A \) and \( C(v) \subset A \). This shows \( DCF^\delta_{\kappa \lambda} \subset F^\delta_{\kappa \lambda} \).

The next theorem shows that the \( F^\delta_{\kappa \lambda} \) form a strictly increasing hierarchy.

Theorem 11. \( F^\delta_{\kappa \lambda} \subset F^{\delta^+}_{\kappa \lambda} \), for all regular \( \delta < \delta^+ \).

Proof. Define \( v : P_{\delta^+} \lambda \rightarrow P_\kappa \lambda \) as follows:
\[
v(x) = \{ \sup x \}.
\]
The result will follow from Theorem 10 once it is demonstrated that no unbounded subset of \( C(v) \) can be closed under \( \delta \)-directed sets. Let \( A \subset C(v) \) be an unbounded subset. Construct \( \{ \alpha_\gamma : \gamma < \delta \} \) and \( \{ x_\gamma : \gamma < \delta \} \) as follows:
Let \( \alpha_1 \in \lambda \), choose \( x_1 \in A \) such that \( \alpha_1 < x_1 \). If \( \xi < \gamma \) and we have \( \alpha_\xi \in \lambda \) and \( x_\xi \in A \) defined, choose
\[
\alpha_\gamma > \sup \{ \bigcup x_\xi : \xi < \gamma \}.
\]
Now choose \( x_\gamma \in A \) such that \( \alpha_\gamma \in x_\gamma \) and \( x_\xi \subset x_\gamma \) when \( \xi < \gamma \). This shows that
\[
\{ x_\gamma : \gamma < \delta \} \text{ is } \delta \text{-directed,}
\]
\[
|\{ x_\gamma : \gamma < \delta \}| < \kappa,
\]
and
\[
\{ x_\gamma : \gamma < \delta \} \subset A.
\]
Suppose \( \bigcup\{ x_\gamma : \gamma < \delta \} \in A \). Consider \( x = \{ \alpha_\gamma : \gamma < \delta \} \). Then
\[
x \in P_\delta \left( \bigcup\{ x_\gamma : \gamma < \delta \} \right)
\]
and
\[
\bigcup\{ x_\gamma : \gamma < \delta \} \in C(v)
\]
since \( A \subset C(v) \). So
\[
v(x) = \{ \sup x \} \subset \bigcup\{ x_\gamma : \gamma < \delta \}.
\]
But clearly \( \sup x \notin \bigcup\{ x_\gamma : \gamma < \delta \} \), a contradiction. \( \square \)

The next set of definitions and theorem will be used to establish that, at each level of the hierarchy, \( F^\delta_{\kappa \lambda} \) is the minimal \( \delta \)-normal filter on \( P_\kappa \lambda \).
Definition 12. For any filter \( F \) on \( P_\kappa \lambda \), denote
\[
\Delta F = \{ A \subseteq P_\kappa \lambda : \exists \{ A_\alpha : \alpha < \lambda \} \subseteq F \text{ and } \Delta_{\alpha < \lambda} A_\alpha \subseteq A \}.
\]

In [Ca], Carr determined that for every filter \( F \) on \( P_\kappa \lambda \) extending \( FSF_{\kappa \lambda} \):

1. \( F \subseteq \Delta F \), where equality holds if \( F \) is normal;
2. \( \Delta FSF_{\kappa \lambda} = SCF_{\kappa \lambda} = (F_{\kappa \lambda}^2) \);
3. \( \Delta \Delta FSF_{\kappa \lambda} = \Delta SCF_{\kappa \lambda} = CF_{\kappa \lambda} = F_{\kappa \lambda}^3 = F_{\kappa \lambda}^{o \alpha} \).

Definition 13. For any filter \( F \) on \( P_\kappa \lambda \), let
\[
\Delta \delta F = \{ A \subseteq P_\kappa \lambda : \exists \{ A_\alpha : \alpha < \lambda \} \subseteq F \text{ and } \Delta_{\alpha < \lambda} A_\alpha \subseteq A \}.
\]

Theorem 14. For regular \( \delta < \delta' \),

1. \( \Delta \delta FSF_{\kappa \lambda} = F_{\kappa \lambda}^\delta \);
2. \( \Delta \delta F_{\kappa \lambda}^\xi = F_{\kappa \lambda}^\delta \) for all regular \( \xi < \delta' \).

Proof. For the proof of part one, let \( C \in \Delta \delta FSF_{\kappa \lambda} \). Without loss of generality, assume
\[
C = \Delta \delta \{ \tilde{x}_y : y \in P_\delta \lambda \},
\]
where \( x_y \in P_\kappa \lambda \) for \( y \in P_\delta \lambda \). Define \( v_y : P_\delta \lambda \rightarrow P_\kappa \lambda \) by \( v_y(z) = x_y \). Given
\[
x \in \Delta \{ C(v_y) : y \in P_\delta \lambda \} \in F_{\kappa \lambda}^\delta.
\]

If \( y \in P_\delta \lambda \), then \( x \in C(v_y) \). So \( x_y = v_y(z) \subseteq x \) for every \( z \in P_\delta \lambda \). Hence \( x \in \tilde{x}_y \), giving \( x \in C \) and therefore \( C \in F_{\kappa \lambda}^\delta \).

Next, let \( v : P_\delta \lambda \rightarrow P_\kappa \lambda \) and consider \( C(v) \). For each \( y \in P_\delta \lambda \), let \( x_y \in C(v) \) such that \( y \subseteq x_y \). Given \( x \in \Delta \{ \hat{x}_y : y \in P_\delta \lambda \} \), let \( y \in P_\delta \lambda \). Then \( x \in \hat{x}_y \) so \( y \subseteq x_y \subseteq x \). Hence \( v(y) \subseteq x_y \subseteq x \). Therefore, \( x \in C(v) \) and \( C(v) \in \Delta \delta FSF_{\kappa \lambda} \).

For the proof of part two, let \( C \in \Delta \delta F_{\kappa \lambda}^\xi \), for any regular \( \xi < \delta' \). Without loss of generality, assume
\[
C = \Delta \delta \{ C(v_y) : y \in P_\delta \lambda \} \text{ and } v_y : P_\delta \lambda \rightarrow P_\kappa \lambda \text{ by } v_y(z) = x_y.
\]

Define \( v_y : P_\delta \lambda \rightarrow P_\kappa \lambda \) by \( v_y(s) = \bigcup \{ v_y(t) : t \in P_\delta s \} \). Now
\[
\Delta \delta \{ C(v_y) : y \in P_\delta \lambda \} \in F_{\kappa \lambda}^\delta,
\]

since \( F_{\kappa \lambda}^\delta \) is \( \delta \)-normal. Given \( x \in \Delta \delta \{ C(v_y) : y \in P_\delta \lambda \} \), \( y \in P_\delta \lambda \) and \( t \in P_\delta x \), then for any \( s \in P_\delta x \) such that \( t \subseteq s \), yields \( v_y(t) \subseteq v_y(s) \subseteq x \). Hence \( x \in C \). Therefore, \( C \in F_{\kappa \lambda}^\delta \).

Finally, let \( v : P_\delta \lambda \rightarrow P_\kappa \lambda \). For \( y \in P_\delta \lambda \), define \( v_y : P_\delta \lambda \rightarrow P_\kappa \lambda \) by \( v_y(t) \in C(v) \) such that \( y \subseteq v_y(t) \) for every \( t \in P_\delta \lambda \). This gives \( C(v_y) \in F_{\kappa \lambda}^\delta \) for each \( y \in P_\delta \lambda \). Let \( x \in \Delta \delta \{ C(v_y) : y \in P_\delta \lambda \} \). If \( y \in P_\delta \lambda \), then \( x \in C(v_y) \). For any \( t \in P_\delta x \), it is the case that \( y \subseteq v_y(t) \) and \( v_y(t) \in C(v) \). This gives \( v(y) \subseteq v_y(t) \). Therefore \( x \in C(v) \) and \( C(v) \in \Delta \delta F_{\kappa \lambda}^\xi \).

Corollary 15. \( F_{\kappa \lambda}^\delta \) is the minimal \( \delta \)-normal, fine, \( \kappa \)-complete filter on \( P_\kappa \lambda \).

Proof. Let \( F \) be any \( \delta \)-normal, fine, \( \kappa \)-complete filter on \( P_\kappa \lambda \). Then \( FSF_{\kappa \lambda} \subseteq F \) and \( \Delta \delta FSF_{\kappa \lambda} \subseteq \Delta \delta F = F \), giving \( F_{\kappa \lambda}^\delta \subseteq F \). □
This section looks at the notions of stationarity and saturation related to the hierarchy of filters, $F^{\delta}_{\kappa \lambda}$ for $\delta < \delta$ regular and presents some well-known results in this context.

Assume $\kappa$ is inaccessible for the remainder of this section. The filters, $DCF^{\delta}_{\kappa \lambda}$ for $\delta < \delta$ and their dual ideals $(DCF^{\delta}_{\kappa \lambda})^*$, are natural extensions of $CF_{\kappa \lambda}$ and the nonstationary ideal, $NS_{\kappa \lambda}$. Since $F^{\delta}_{\kappa \lambda} = DCF^{\delta}_{\kappa \lambda}$ it holds that $(F^{\delta}_{\kappa \lambda})^+ \subset CF^{\delta}_{\kappa \lambda}$.

**Definition 16.** Say $S \subseteq P_\kappa \lambda$ is $\delta$-stationary if $A \cap S \neq \emptyset$, for all $A$ in $F^{\delta}_{\kappa \lambda}$. That is, $S$ is $\delta$-stationary if and only if $S \subseteq (F^{\delta}_{\kappa \lambda})^+$.

The following theorem is in the spirit of [Me, Corollary 16], and characterizes $\delta$-stationary sets in terms of “$\delta$-regressive” functions.

**Theorem 17.** For $\delta < \delta$, $S \subseteq P_\kappa \lambda$ is $\delta$-stationary if and only if for every $f : P_\kappa \lambda \rightarrow P_\lambda$ such that $f(x) \subseteq P_\delta x$, $\forall x \in S$ (say $f$ is $\delta$-regressive over $S$), there exists an unbounded subset of $S$ where $f$ is constant.

**Proof.** One direction follows directly from the fact that $F^{\delta}_{\kappa \lambda}$ is $\delta$-normal. For the other direction, assume that $S$ is not $\delta$-stationary. Let $v : P_\delta \lambda \rightarrow P_\kappa \lambda$ be such that $C(v) \subseteq P_\delta \lambda - S$. Define $f : S \rightarrow P_\delta \lambda$ by $f(x) = P_\delta x$ such that $v(f(x)) \not\subseteq x$. But this leads to a contradiction, since $f^{-1} \{ \{y\} \} \cap S$ cannot be unbounded for any $y \in P_\delta \lambda$. □

Recall that an ideal $I$ is said to be $\mu$-saturated if whenever $\{ A_\gamma : \gamma < \mu \} \subseteq I^+$ such that $A_\gamma \cap A_{\gamma'} \in I$ for $\gamma \neq \gamma'$, then $\nu < \mu$.

The next theorem gives a characterization of saturation for $\delta$-normal ideals on $P_\kappa \lambda$ in terms of their possible extensions by $\delta$-normal ideals on $P_\kappa \lambda$. This type of characterization was first used in [B-T-W] for normal ideals on a cardinal $\kappa$.

Given an ideal $I$ on $P_\kappa \lambda$, let

$$I|A = \{ B \subseteq P_\kappa \lambda : A \cap B \in I \}.$$ 

This is the ideal generated by $I$ and $P_\kappa \lambda - A$. Observe that if $A \in I^+$, then $I|A = I$ and if $A \in I$ then $I|A = P(P_\kappa \lambda)$. Hence $I|A$ is a proper extension of $I$ when $A \in I^+ - I^*$.

**Theorem 18.** Let $\delta < \kappa$. A $\delta$-normal, fine ideal $I$ on $P_\kappa \lambda$ is $(\lambda^{< \delta})^+$-saturated if and only if $I|A$ for $A \in I^+ - I^*$ are the only $\delta$-normal ideals on $P_\kappa \lambda$ which properly extend $I$.

**Proof.** The proof follows the argument of [B-T-W] with straightforward modifications. □

**Mahlo’s operation:** for $A \subseteq \kappa$,

$$M(A) = \{ \alpha \in \kappa : \text{cf} (\alpha) > \omega \text{ and } A \cap \alpha \notin NS_\alpha \}$$

led to the notion of an $M$-ideal, which plays a major role in gaining results on splitting stationary sets on $\kappa$ (see [B-T-W]), and on $P_\kappa \lambda$ (see [Jo]). In the latter case, Johnson first generalized the Mahlo operation to $P_\kappa \lambda$ as follows: for $A \subseteq P_\kappa \lambda$

$$M(A) = \{ x \in A : x \cap \kappa \text{ is a weak. inacc. and } A \cap P_{x \cap \kappa} x \in NS^+_{x \cap \kappa}, x \}.$$
An $M$-ideal is an ideal $I$, which satisfies: $M(A) \in I^*$ if and only if $A \in I^*$.

Theorem 18, in combination with the following statement provide the base for the results on splitting stationary subsets in the two papers mentioned in the preceding paragraph: If $I$ is an $M$-ideal then, for any stationary subset $A$, $I \neq NS|A$ (where $NS$ stands for the nonstationary ideal on $\kappa$ or $P_\kappa \lambda$). In the latter case, the fact that the closed unbounded filter on $P_\kappa \lambda$ can be replaced by the filter generated by closure sets of functions $v : P_\delta \lambda \rightarrow P_\kappa \lambda$ provides the key to the proof of this statement.

A variation of Johnson's generalization of the Mahlo operation to $P_\kappa \lambda$ is presented, which is directly analogous to the way Jech generalized the diamond principle to $P_\kappa \lambda$ in [Je2].

**Definition 19.** For $A \subseteq P_\kappa \lambda$ and $\delta < \delta$,

$$M^\delta(A) = \{ x \in A : |x| \text{ is weak, inacc., } |x| > \delta \text{ and } A \cap P_{|x|} \lambda \in (F^\delta_{|x|, \lambda})^+ \}.$$  

And for $I$ a $\delta$-normal ideal, say $I$ is an $M^\delta$-ideal if $M^\delta(A) \in I^*$ whenever $A \in I^*$.

The usefulness of the filter generated by closure sets of functions $v : P_\delta \lambda \rightarrow P_\kappa \lambda$ will be used to extend the statement mentioned in the preceding paragraph to this context.

**Theorem 20.** If $I$ is an $M^\delta$-ideal on $P_\kappa \lambda$ then, for any $A \in (F^\delta_{\kappa \lambda})^+$, $I \neq (F^\delta_{\kappa \lambda})^*|A$.

**Proof.** Suppose $I$ is an $M^\delta$-ideal and $I = (F^\delta_{\kappa \lambda})^*|A$ for some $A \in (F^\delta_{\kappa \lambda})^+$. Then there exists a $v : P_\delta \lambda \rightarrow P_\kappa \lambda$ such that $C(v) \cap A \subseteq M^\delta(A)$.

Claim. There exists a $B \in C(v)$ such that

$$B = \{ x \in P_\kappa \lambda : \forall y \in P_\delta x( v(y) \subseteq x \land |v(y)| < |x|) \}$$

and $B \in DCF^{\delta}_{\kappa \lambda}$.

**Proof of claim.** The proof of this claim is a straightforward exercise and will be omitted.

Let $x \in B \cap A \subseteq M^\delta(A)$ be such that $|x|$ is minimal. Define $v_x : P_\delta x \rightarrow P_{|x|} x$ by $v_x(y) = v(y)$. This holds since $x \in C(v)$ and $y \in P_\delta x$, and hence $v_x(y) \subseteq x$. Also, $x \in B$ gives $|v_x(y)| < |x|$ and $v_x(y) \in P_{|x|} x$. But $C(v_x) \subseteq C(v) \cap P_{|x|} x$, since first: $v_x$ assumes values in $P_{|x|} x$ so $C(v_x) \subseteq P_{|x|} x$; and if $y \in C(v_x)$ and $z \in P_\delta y \subseteq P_\delta x$, then $v(z) = v_x(z) \subseteq y$; hence, $y \in C(v)$. Next, let $B_x \subseteq C(v_x)$ such that

$$B_x = \{ y \in P_{|x|} x : \forall z \in P_\delta y( v_x(z) \subseteq y \land |v_x(z)| < |y|) \}.$$

Now $B_x \in F^\delta_{|x|, \lambda}$ and hence $B \cap P_{|x|} x \subseteq F^\delta_{|x|, \lambda}$. But $x \in M^\delta(A)$, so $A \cap P_{|x|} x \in (F^\delta_{|x|, \lambda})^+$. But this means that there exists $y \in B \cap A \cap P_{|x|} x$, contradicting the choice of $x$. □

The theorems of this section are not meant to be exhaustive, but rather they are meant to suggest that a great deal of what can be done for $C_{\kappa \lambda}$ can in turn be done for $F^\delta_{\kappa \lambda}$.

A consequence of Matsubara's technique is that $P_\kappa \lambda$ splits into $\lambda^{<\kappa}$ many disjoint $\delta$-stationary subsets.
References


MATHEMATICAL SCIENCES RESEARCH INSTITUTE, BERKELEY, CALIFORNIA 94720

Current address: Department of Mathematics, The College of Charleston, Charleston, South Carolina 29424