A CHARACTER SUM FOR ROOT SYSTEM $G_2$

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Abstract. A character sum analog of the Macdonald-Morris constant term identity for the root system $G_2$ is proved. The proof is based on recent evaluations of Selberg character sums and on a character sum analog of Dixon's summation formula. A conjectural evaluation is presented for a related sum.

1. Introduction

Let $GF(q)$ denote the finite field of $q$ elements, where $q$ is a power of an odd prime $p$. Throughout, $A$, $B$, and $C$ denote multiplicative characters on $GF(q)$. Let $1$ and $\phi$ denote the trivial and quadratic characters on $GF(q)$, respectively. Define $A(0) = 0$, even if $A = 1$. Let $\text{ord} C$ denote the order of $C$ (e.g., $\text{ord} \phi = 2$).

Define the Gauss and Jacobi sums $G(A)$, $J(A, B)$ over $GF(q)$ by

$$
G(A) = \sum_{m \in GF(q)} A(m)\zeta^{T(m)}, \quad J(A, B) = \sum_{m} A(m)B(1 - m),
$$

where the sums are over all $m \in GF(q)$, $\zeta = \exp(2\pi i/p)$, and $T$ denotes the trace map from $GF(q)$ to $GF(p)$. For nonnegative integers $n$, define the $n$-dimensional Selberg character sum $L_n(A, B, C)$ over $GF(q)$ by

$$
L_n(A, B, C) = \sum_{\text{deg } F = n} A((-1)^n F(0))B(F(1))C\phi(D_F),
$$

where the sum is over all monic polynomials $F$ over $GF(q)$ of degree $n$, and where $D_F$ denotes the discriminant of $F$.

Define

$$
R_n(A, B, C) = \prod_{j=0}^{n-1} \frac{G(C^{j+1})G(AC^{j})G(BC^{j})}{G(C)G(ABC^{n-1+j})}
$$

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and
\[
S_n(A, B, C) = q^{-n} R_n(A, B, C) \prod_{j=0}^{n-1} |G(ABC^{n-1+j})|^2
\]
\[
= q^{-n} G(C)^{-n} \prod_{j=0}^{n-1} G(C^{j+1})G(AC^j)G(BC^j)G(ABC^{n-1+j}).
\]

The generic Selberg character sum formula in Theorem 1.1 was conjectured in [5, (2.6); 2, (29)]. A proof of Theorem 1.1, based on the method of Anderson [1], is given in [4].

**Theorem 1.1.** If

\[ ABC^{n-1+j} \text{ is nontrivial for all } j, \quad 0 \leq j \leq n - 1 \]

or

\[ AC^a \text{ is nontrivial for all } a, \quad 0 \leq a \leq n - 1 \]

or

\[ BC^b \text{ is nontrivial for all } b, \quad 0 \leq b \leq n - 1, \]

then

\[ L_n(A, B, C) = S_n(A, B, C). \]

Using Theorem 1.1 we prove a character sum analog of the Macdonald-Morris constant term identity for the root system $G_2$ [9, p. 994; 10, p. 45]. This analog, given in Theorem 1.2, was inspired by a pretty paper of Zeilberger [11].

**Theorem 1.2.** Let

\[ L = \sum_{\substack{F(0)=-1 \deg F=3}} B^2(F(1))C\phi(D_F), \]

where the sum is over all monic cubic polynomials $F$ over $GF(q)$ with constant term $-1$. Then

\[ L = q^2 - 2q + 3, \quad \text{if } B^2 = 1, \ C = \phi, \]

\[ L = (2 - 4/q)G(C)^3, \quad \text{if } B^2 = 1, \ \text{ord } C = 3, \]

\[ L = (1 - 3/q)G(C)^3, \quad \text{if } B^2 = C^2, \ \text{ord } C = 3, \]

and

\[ L = P(B, C) + P(B\phi, C) \quad \text{otherwise}, \]

where

\[ P(B, C) = \frac{G(C^2)G(C^3)G(B^2)G(B^2C^3)G(BC^2)G(B^3C^3)}{G(B)G(BC)G(C)^2}. \]

Note the completely direct analogy between $P(B, C)$ and the product of gamma functions in the Macdonald-Morris identity for $G_2$. The form of the sum $L$ in (1.9) is suggested by identifying the polynomial $F(W)$ in (1.9) with $(W-x/y)(W-y/z)(W-z/x)$, where $x$, $y$, $z$ are the variables in the constant
term identity for $G_2$ in [11, Theorem, p. 880]. The form of the sum $L$ is not directly analogous to the trigonometric integral [10, p. 46] or the beta integral [6, (1.7)] associated with $G_2$.

We remark that if $B^2$ is replaced by a nonsquare character in (1.9), then the resulting sum vanishes. This follows from (2.1) below and [5, (2.2)].

Our proof of Theorem 1.2 employs the character sum analog of Dixon’s summation formula [11, p. 881] given in Theorem 1.3. A proof of this analog (and more general results) can be found in [7]; we give a different proof in the Appendix.

**Theorem 1.3.** Define

\[
\delta(A) = \begin{cases} 
0, & \text{if } A \neq 1, \\
1, & \text{if } A = 1.
\end{cases}
\]

Then for all characters $D$, $E$, $F$ on $GF(q)$,

\[
(q - 1)^{-1} \sum_{A} G(AD)G(AE)G(AF)\overline{G}(A\overline{D})\overline{G}(A\overline{E})\overline{G}(A\overline{F})
\]

\[
= (q - 1)q^2\delta(DE^2F^2) + Q(D, E, F) + Q(D\phi, E\phi, F\phi),
\]

where

\[
Q(D, E, F) = DEF(-1)G(DE)G(DF)G(EF)G(D)G(E)G(F)/G(DEF).
\]

Our proof of Theorem 1.2 also requires the evaluations of the Selberg sums $L_3(\overline{C}^2, 1, C\phi)$ and $L_3(\overline{C}, \overline{C}, C\phi)$ given in Theorem 1.4. These two Selberg sums are not covered by Theorem 1.1, but they can be evaluated by a suitable modification of the proof of [4, Theorem 1.1]. We omit the details.

**Theorem 1.4.** If $C^2 \neq 1$, then

\[
\frac{L_3(\overline{C}^2, 1, C\phi)}{R_3(\overline{C}^2, 1, C)} = \frac{L_3(\overline{C}, \overline{C}, C\phi)}{R_3(\overline{C}, \overline{C}, C)} = 2 - q.
\]

Inspired by Theorem 1.2, Greg Anderson suggested that the sum

\[
Y(B, C) := \sum_{x, y \in GF(q)} B(x^2 - 4y)C(y^2 + 18y + 12xy - 4x^3 - 27)
\]

has an elegant product formula. Since the discriminant of the polynomial $F(z) = z^3 - rz^2 + sz - 1$ is $r^2s^2 + 18rs - 4s^3 - 4r^3 - 27$, one sees via the transformation $x = r + s$, $y = rs$ that

\[
L = Y(B, C) + Y(B\phi, C\phi).
\]

Thus the following conjecture implies Theorem 1.2.

**Conjecture 1.5.** We have

\[
Y(B\phi, C\phi) = q^2 - 2q + 2 = (q^2 - 2q + 2)P(B, C), \quad \text{if } B = C = \phi,
\]

\[
Y(B\phi, C\phi) = (1 - 2/q)G(\overline{C})^3 = (2 - q)qP(B, C), \quad \text{if } \text{ord } C = 3, \quad B \in \{1, \phi, C\},
\]

\[
L = Y(B, C) + Y(B\phi, C\phi).
\]
and

\[(1.23) \quad Y(B\phi, C\phi) = P(B, C), \quad \text{otherwise.} \]

For character sum analogs of Macdonald-Morris constant term identities connected with various other root systems, see [3]. For most root systems (e.g., $F_4$, $E_6$, $E_7$, $E_8$, ...), no analogs are known.

## 2. Proof of Theorem 1.2

By (1.2) and (1.9),

\[(2.1) \quad L = \frac{1}{q-1} \sum_A L_3(A, B^2, C\phi). \]

Define

\[(2.2) \quad d(A, B, C) = L_3(A, B, C\phi) - S_3(A, B, C). \]

Then by (2.1) and Theorem 1.1,

\[(2.3) \quad L = T + \frac{1}{q-1} \sum_{A \in \{1, C, C^2\}} d(A, B^2, C), \]

where

\[(2.4) \quad T = \frac{1}{q-1} \sum_A S_3(A, B^2, C). \]

By (2.4) and (1.4),

\[(2.5) \quad T = \frac{1}{q-1} \sum_A S_3(ABC^2, B^2, C) \]

Define

\[(2.6) \quad D = BC^2, \quad E = BC, \quad F = B \]

to obtain, for all characters $B, C$,

\[(2.7) \quad T = \frac{G(C^2)G(C^3)G(B^2)G(B^2C)G(B^2C^2)}{q^3G(C)^2} \sum_A \prod_{j=0}^{2} G(ABC^{-j})G(ABC^j). \]

Apply Theorem 1.3 with

\[(2.8) \quad Q(BC^2, BC, B) \]

By definition (1.17),

\[(2.9) \quad W(B, C) = G(B^2C^3)G(B^2C^2)G(B^2C)G(BC^2)G(B^C)G(B)/G(B^3C^3). \]
By (2.8) and (2.9),
\[(2.10)\]
\[W(B, C) = Q(\overline{BC^2}, \overline{BC}, \overline{B}), \quad \text{if } B^3C^3 \neq 1.\]

Assume first that
\[(2.11)\]
\[B^2, B^2C, \text{ and } B^2C^2 \text{ are nontrivial.}\]

By (2.11), if \(B^3C^3 = 1\), then
\[(2.12)\]
\[W(B, C) = -q^2 \quad \text{and} \quad Q(\overline{BC^2}, \overline{BC}, \overline{B}) = -q^3.\]

Hence (2.10) has the extension
\[(2.13)\]
\[(q - 1)q^2\delta(B^3C^3) + Q(\overline{BC^2}, \overline{BC}, \overline{B}) = W(B, C).\]

Since \(\delta(B^6C^6) = \delta(B^3C^3) + \delta(\phi B^3C^3)\), the expression in braces in (2.7) equals
\[(2.14)\]
\[W(B, C) + W(B\phi, C).\]

Again using (2.11), we thus obtain
\[(2.15)\]
\[T = P(B, C) + P(B\phi, C).\]

By (2.11) and Theorem 1.1, each summand \(d(\overline{C^a}, B^2, C)\) in (2.3) vanishes. Thus \(L = T\) and the result follows from (2.15) under the assumption (2.11).

Now drop the assumption (2.11). For brevity, set
\[(2.16)\]
\[R(a, b) = R(\overline{C^a}, \overline{C^b}, C),\]
\[(2.17)\]
\[U(a, b) = L_3(\overline{C^a}, \overline{C^b}, C\phi)/R(a, b),\]
\[(2.18)\]
\[V(a, b) = S_3(\overline{C^a}, \overline{C^b}, C)/R(a, b),\]

where \(0 \leq a, b \leq 2\). Observe that \(R(a, b), U(a, b), V(a, b)\) are symmetric in \(a, b\). We proceed to evaluate these functions.

From (1.3),
\[(2.19)\]
\[R(0, 0) = G^2(C^2)/G(C^4),\]
\[(2.20)\]
\[R(1, 0) = G(\overline{C})G(C^2)/G(C),\]
\[(2.21)\]
\[R(2, 0) = R(2, 2) = R(2, 1) = -|G(C^2)|^2G(C^3)G(\overline{C})/G^2(C),\]
\[(2.22)\]
\[R(1, 1) = -G(C^3)G^2(\overline{C})/G(C).\]

From (1.4),
\[(2.23)\]
\[V(0, 0) = \begin{cases} q^{-3}, & \text{if } C = 1, \\ q^{-2}, & \text{if } C = \phi, \\ q^{-1}, & \text{if ord } C = 3 \text{ or } 4, \\ 1, & \text{if ord } C > 4, \end{cases}\]
\[(2.24)\]
\[V(1, 0) = \begin{cases} q^{-3}, & \text{if } C = 1, \\ q^{-1}, & \text{if ord } C = 2 \text{ or } 3, \\ 1, & \text{if ord } C > 3, \end{cases}\]
(2.25) \[ V(2, 0) = V(2, 2) = V(1, 1) = \begin{cases} q^{-3}, & \text{if } C = 1, \\ q^{-2}, & \text{if } C = \phi, \\ q^{-1}, & \text{if } \text{ord } C > 2, \end{cases} \]

(2.26) \[ V(2, 1) = \begin{cases} q^{-3}, & \text{if } C = 1, \\ q^{-1}, & \text{if } C \neq 1. \end{cases} \]

By [5, Theorem 4.1],

(2.27) \[ U(0, 0) = \begin{cases} 4 - 3q, & \text{if } C = 1, \\ -q^3 + 3q^2 - 5q + 4, & \text{if } C = \phi, \\ q^2 - 3q + 3, & \text{if } \text{ord } C = 3, \\ q^{-1}, & \text{if } \text{ord } C = 4, \\ 1, & \text{if } \text{ord } C > 4. \end{cases} \]

We claim that

(2.28) \[ U(1, 0) = \begin{cases} 4 - 3q, & \text{if } C = 1, \\ 2 - q, & \text{if } C = \phi, \\ q^2 - 3q + 3, & \text{if } \text{ord } C = 3, \\ 1, & \text{if } \text{ord } C > 3. \end{cases} \]

The cases \( C = 1, C = \phi \) of (2.28) follow from [5, (2.13), (2.14)]. The case where \( \text{ord } C = 3 \) follows from (2.27), since by [5, Lemmas 2.1, 2.2],

\[ U(1, 0) = U(0, 0) \quad \text{if } \text{ord } C = 3. \]

The last case where \( \text{ord } C > 3 \) follows from (2.24) and Conjecture 1.1 (note that the hypothesis (1.5) of Theorem 1.1 holds with \( A = C \), \( B = 1 \)). Next we claim that

(2.29) \[ U(2, 0) = U(2, 2) = \begin{cases} 4 - 3q, & \text{if } C = 1, \\ -q^3 + 3q^2 - 5q + 4, & \text{if } C = \phi, \\ 2 - q, & \text{if } \text{ord } C > 2. \end{cases} \]

The first equality in (2.29) follows from [5, Lemmas 2.1, 2.2]. The cases \( C = 1, C = \phi \) of (2.29) follow from [5, (2.13), (2.14)], while the remaining case follows from Theorem 1.4. The same argument shows that

(2.30) \[ U(1, 1) = \begin{cases} 4 - 3q, & \text{if } C = 1, \\ (2 - q)/q, & \text{if } C = \phi, \\ 2 - q, & \text{if } \text{ord } C > 2. \end{cases} \]

Finally, we claim that

(2.31) \[ U(2, 1) = \begin{cases} 4 - 3q, & \text{if } C = 1, \\ 2 - q, & \text{if } C \neq 1. \end{cases} \]

The cases \( C = 1, C = \phi \) of (2.31) follow from [5, (2.13), (2.14)], while the cases where \( C^2 \neq 1 \) follow from (2.30), since

(2.32) \[ U(2, 1) = U(1, 1) \quad \text{if } \text{ord } C > 2 \]

by [5, Lemmas 2.1, 2.2].
For $0 \leq a, b \leq 2$, set
\begin{equation}
(2.33)
    d(a, b) = \{(U(a, b) - V(a, b))R(a, b),
\end{equation}
so that by (2.2),
\begin{equation}
(2.34)
    d(a, b) = d(\overline{e}^a, \overline{e}^b, C).
\end{equation}
From (2.19)–(2.31), we obtain the following evaluation of $d(a, b)$:
\begin{equation}
(2.35)
    d(0, 0) = \begin{cases}
        (4-3a-q^{-3}), & \text{if } C = 1, \\
        (3a^3 + 3a^2 - 5q + 4 - q^{-2}), & \text{if } C = \phi, \\
        (q^2 - 3a + 3 - q^{-1})G^3(\overline{C})/q, & \text{if ord } C = 3, \\
        0, & \text{if ord } C > 3;
    \end{cases}
\end{equation}
\begin{equation}
(2.36)
    d(1, 0) = \begin{cases}
        (4-3q-q^{-3}), & \text{if } C = 1, \\
        (q^2 + 3q^2 - 5q + 4 - q^{-2}), & \text{if } C = \phi, \\
        (q^2 - 3q + 3 - q^{-1})G^3(\overline{C})/q, & \text{if ord } C = 3, \\
        0, & \text{if ord } C > 3;
    \end{cases}
\end{equation}
\begin{equation}
(2.37)
    d(2, 0) = d(2, 2) = \begin{cases}
        (4-3a-q^{-3}), & \text{if } C = 1, \\
        (q^2 + 3q^2 - 5q + 4 - q^{-2}), & \text{if } C = \phi, \\
        (2a^3 - q^{-1})G^3(\overline{C})/q, & \text{if ord } C > 2;
    \end{cases}
\end{equation}
\begin{equation}
(2.38)
    d(1, 1) = \begin{cases}
        (4-3a-q^{-3}), & \text{if } C = 1, \\
        (q^2 + 3q^2 - 5q + 4 - q^{-2}), & \text{if } C = \phi, \\
        (2a^3 - q^{-1})G^3(\overline{C})/q, & \text{if ord } C > 2;
    \end{cases}
\end{equation}
and
\begin{equation}
(2.39)
    d(2, 1) = \begin{cases}
        (4-3a-q^{-3}), & \text{if } C = 1, \\
        (q^2 + 3q^2 - 5q + 4 - q^{-2}), & \text{if } C = \phi, \\
        (2a^3 - q^{-1})G^3(\overline{C})/q, & \text{if ord } C > 2.
    \end{cases}
\end{equation}
We now evaluate $L$ from (2.3), using (2.7), (2.8), and (2.35)–(2.39), and Theorem 1.2 follows.

3. APPENDIX

Here we give a proof of Theorem 1.3. Let $H$ denote the left side of (1.16). First suppose that $DE = 1$. Then
\begin{equation}
(3.1)
    H = \frac{1}{q-1} \sum_A G(AE)^2 G(A\overline{E})^2 G(AF)G(\overline{A}F),
\end{equation}
\begin{equation}
(3.2)
    M = \frac{q^2}{q-1} \sum_A G(AF)G(\overline{A}F).
\end{equation}
By (1.1),

\[ M = \frac{q^2}{q-1} \sum_t \sum_u \sum_A A(t)A(u)\zeta^{T(t-u)} = q^2(q - 1)\delta(F^2). \]

Using (3.3) in (3.1), we easily deduce (1.16) in the case $DE = 1$.

By symmetry, it remains to prove (1.16) in the case

\[ DE \neq 1, \quad DF \neq 1, \quad EF \neq 1. \]

By (1.1),

\[ \Delta = -\frac{4}{q-1} \sum_{t,u,v} x,y,z \neq 0 \sum_A A(tuv)D(tx)E(uy)F(vz)\zeta^{T(t+u+v-x-y-z)} \]

\[ = -\frac{1}{q-1} \sum_{t,u,v} x,y,z \neq 0 \sum_A A(tuv)D(tx)E(uy)F(vz)\zeta^{T(y(t-1)+z(u-1)+x(v-1))}, \]

where the last equality results from replacing $t$ by $ty$, $u$ by $uz$, and $v$ by $vx$. By (3.4), it follows that

\[ H = -\frac{1}{q-1} G(DE)G(DF)G(EF) \]

\[ \times \sum_{t,u,v} A(tuv)DE(1-t)EF(1-u)DF(1-v)D(t)E(u)F(v). \]

Thus,

\[ H/\{G(DE)G(DF)G(EF)\} \]

\[ = \sum_{t,v \neq 0} DE(1-t)EF(1-1/(tv))DF(1-v)D(t)E(tv)F(v) \]

\[ = \sum_{t,v \neq 0} DE(1-t/v)EF(1-1/t)DF(1-v)D(t/v)E(t)F(v) \]

\[ = \sum_{t,v} EF(v)DF(t)EF(t-1)DF(1-v)DE(v-t) \]

\[ = EF(-1) \sum_{t,v \neq 0} DF \left(1 + \frac{v}{t}\right) EF \left(1 + \frac{t}{v}\right) DE(v-t) \]

\[ = EF(-1)\{J(\overline{DE}, DE)J(E, \overline{DE}) + J(\overline{DE}, \phi), DE)J(E\phi, \overline{DE})\}, \]

where the last equality follows from [2, (28)]. Since $DE \neq 1$, we can apply the formula [8]

\[ J(A, B) = G(A)G(\overline{AB})A(-1)/G(\overline{B}), \quad \text{if } B \neq 1 \]

to express all of the Jacobi sums in (3.7) in terms of Gauss sums. Then (1.16) readily follows.

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