A CHARACTER SUM FOR ROOT SYSTEM \textit{G}_2

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\textbf{Abstract.} A character sum analog of the Macdonald–Morris constant term identity for the root system \textit{G}_2 is proved. The proof is based on recent evaluations of Selberg character sums and on a character sum analog of Dixon's summation formula. A conjectural evaluation is presented for a related sum.

1. \textbf{Introduction}

Let \( GF(q) \) denote the finite field of \( q \) elements, where \( q \) is a power of an odd prime \( p \). Throughout, \( A, B, \) and \( C \) denote multiplicative characters on \( GF(q) \). Let 1 and \( \phi \) denote the trivial and quadratic characters on \( GF(q) \), respectively. Define \( A(0) = 0 \), even if \( A = 1 \). Let \( \text{ord } C \) denote the order of \( C \) (e.g., \( \text{ord } \phi = 2 \)).

Define the Gauss and Jacobi sums \( G(A), J(A, B) \) over \( GF(q) \) by

\begin{equation}
G(A) = \sum_{m} A(m)\zeta^{T(m)}, \quad J(A, B) = \sum_{m} A(m)B(1 - m),
\end{equation}

where the sums are over all \( m \in GF(q) \), \( \zeta = \exp(2\pi i/p) \), and \( T \) denotes the trace map from \( GF(q) \) to \( GF(p) \). For nonnegative integers \( n \), define the \( n \)-dimensional Selberg character sum \( L_n(A, B, C, \phi) \) over \( GF(q) \) by

\begin{equation}
L_n(A, B, C, \phi) = \sum_{\deg F = n} A((-1)^nF(0))B(F(1))C\phi(D_F),
\end{equation}

where the sum is over all monic polynomials \( F \) over \( GF(q) \) of degree \( n \), and where \( D_F \) denotes the discriminant of \( F \).

Define

\begin{equation}
R_n(A, B, C) = \prod_{j=0}^{n-1} \frac{G(C^{j+1})G(AC^{j})G(BC^{j})}{G(C)G(ABC^{n-1+j})}
\end{equation}

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and

\[ S_n(A, B, C) = q^{-n} R_n(A, B, C) \prod_{j=0}^{n-1} |G(ABC^{n-1+j})|^2 \]

(1.4)

\[ = q^{-n} G(C)^{-n} \prod_{j=0}^{n-1} G(C^{j+1}) G(AC^j) G(BC^j) \overline{G(ABC^{n-1+j})}. \]

The generic Selberg character sum formula in Theorem 1.1 was conjectured in [5, (2.6); 2, (29)]. A proof of Theorem 1.1, based on the method of Anderson [1], is given in [4].

**Theorem 1.1.** If

(1.5) \( ABC^{n-1+j} \) is nontrivial for all \( j \), \( 0 \leq j \leq n - 1 \)

or

(1.6) \( AC^a \) is nontrivial for all \( a \), \( 0 \leq a \leq n - 1 \)

or

(1.7) \( BC^b \) is nontrivial for all \( b \), \( 0 \leq b \leq n - 1 \),

then

(1.8) \( L_n(A, B, C \phi) = S_n(A, B, C) \).

Using Theorem 1.1 we prove a character sum analog of the Macdonald-Morris constant term identity for the root system \( G_2 \) [9, p. 994; 10, p. 45]. This analog, given in Theorem 1.2, was inspired by a pretty paper of Zeilberger [11].

**Theorem 1.2.** Let

(1.9) \[ L = \sum_{\text{deg } F = 3} B^2(F(1)) C \phi(D_F), \]

where the sum is over all monic cubic polynomials \( F \) over \( GF(q) \) with constant term \(-1\). Then

(1.10) \( L = q^2 - 2q + 3, \) if \( B^2 = 1, \ C = \phi, \)

(1.11) \( L = (2 - 4/q)G(C)^3, \) if \( B^2 = 1, \ \text{ord } C = 3, \)

(1.12) \( L = (1 - 3/q)G(C)^3, \) if \( B^2 = C^2, \ \text{ord } C = 3, \)

and

(1.13) \( L = P(B, C) + P(B \phi, C) \) otherwise,

where

(1.14) \[ P(B, C) = \frac{G(C^2)G(C^3)G(B^2)G(B^2-C^3)G(BC^2)G(B^3C^3)}{G(B)G(BC)G(C)^2}. \]

Note the completely direct analogy between \( P(B, C) \) and the product of gamma functions in the Macdonald-Morris identity for \( G_2 \). The form of the sum \( L \) in (1.9) is suggested by identifying the polynomial \( F(W) \) in (1.9) with \((W-x/y)(W-y/z)(W-z/x), \) where \( x, y, z \) are the variables in the constant...
term identity for $G_2$ in [11, Theorem, p. 880]. The form of the sum $L$ is not
directly analogous to the trigonometric integral [10, p. 46] or the beta integral
[6, (1.7)] associated with $G_2$.

We remark that if $B^2$ is replaced by a nonsquare character in (1.9), then the
resulting sum vanishes. This follows from (2.1) below and [5, (2.2)].

Our proof of Theorem 1.2 employs the character sum analog of Dixon’s
summation formula [11, p. 881] given in Theorem 1.3. A proof of this analog
(and more general results) can be found in [7]; we give a different proof in the
Appendix.

**Theorem 1.3.** Define

$$(1.15) \quad \delta(A) = \begin{cases} 0, & \text{if } A \neq 1, \\ 1, & \text{if } A = 1. \end{cases}$$

Then for all characters $D, E, F$ on $GF(q),

$$(1.16) \quad (q - 1)^{-1} \sum_A G(AD)G(AE)G(AF)G(AD)G(AE)G(AF) = (q - 1)q^2 \delta(D^2E^2F^2) + Q(D, E, F) + Q(D\phi, E\phi, F\phi),$$

where

$$(1.17) \quad Q(D, E, F) = DEF(-1)G(DEF)G(DF)G(EF)G(D)G(E)G(F)/G(DEF).$$

Our proof of Theorem 1.2 also requires the evaluations of the Selberg sums
$L_3(C^2, 1, C\phi)$ and $L_3(C, C, C\phi)$ given in Theorem 1.4. These two Selberg
sums are not covered by Theorem 1.1, but they can be evaluated by a suitable
modification of the proof of [4, Theorem 1.1]. We omit the details.

**Theorem 1.4.** If $C^2 \neq 1$, then

$$(1.18) \quad \frac{L_3(C^2, 1, C\phi)}{R_3(C^2, 1, C)} = \frac{L_3(C, C, C\phi)}{R_3(C, C, C)} = 2 - q.$$

Inspired by Theorem 1.2, Greg Anderson suggested that the sum

$$(1.19) \quad Y(B, C) := \sum_{x, y \in GF(q)} B(x^2 - 4y)C(y^2 + 18y + 12xy - 4x^3 - 27)$$

has an elegant product formula. Since the discriminant of the polynomial
$F(z) = z^3 - rz^2 + sz - 1$ is $r^2s^2 + 18rs - 4s^3 - 4r^3 - 27$, one sees via the
transformation $x = r + s, \ y = rs$ that

$$(1.20) \quad L = Y(B, C\phi) + Y(B\phi, C\phi).$$

Thus the following conjecture implies Theorem 1.2.

**Conjecture 1.5.** We have

$$(1.21) \quad Y(B\phi, C\phi) = q^2 - 2q + 2 = (q^2 - 2q + 2)P(B, C), \quad \text{if } B = C = \phi,$$

$$(1.22) \quad Y(B\phi, C\phi) = (1 - 2/q)G(C^3) = (2 - q)qP(B, C), \quad \text{if } \text{ord } C = 3, \ B \in \{1, \phi, C\},$$
and

\[(1.23) \quad Y(B\phi, C\phi) = P(B, C), \quad \text{otherwise.}\]

For character sum analogs of Macdonald-Morris constant term identities connected with various other root systems, see [3]. For most root systems (e.g., \(F_4, E_6, E_7, E_8, \ldots\)), no analogs are known.

2. PROOF OF THEOREM 1.2

By (1.2) and (1.9),

\[(2.1) \quad L = \frac{1}{q - 1} \sum_A L_3(A, B^2, C_\phi).\]

Define

\[(2.2) \quad d(A, B, C) = L_3(A, B, C_\phi) - S_3(A, B, C).\]

Then by (2.1) and Theorem 1.1,

\[(2.3) \quad L = T + \frac{1}{q - 1} \sum_{A \in \{1, \bar{C}, \bar{C}^2\}} d(A, B^2, C),\]

where

\[(2.4) \quad T = \frac{1}{q - 1} \sum_A S_3(A, B^2, C).\]

By (2.4) and (1.4),

\[(2.5) \quad T = \frac{1}{q - 1} \sum_A S_3(A\bar{B}C^2, B^2, C)\]
\[= \frac{G(C)G(C^2)G(C^3)G(B^2)G(B^2C)G(B^2C^2)}{(q - 1)q^3G(C)^3} \sum_A \prod_{j=0}^{2} G(A\bar{B}C^{j-2})\bar{G}(ABC^j).\]

Apply Theorem 1.3 with

\[(2.6) \quad D = \bar{B}C^2, \quad E = \bar{B}\bar{C}, \quad F = \bar{B}\]

to obtain, for all characters \(B, C,\)

\[(2.7) \quad T = \frac{G(C^2)G(C^3)G(B^2)G(B^2C)G(B^2C^2)}{q^3G(C)^2}\]
\[\cdot \{(q - 1)q^2\delta(B^6C^6) + Q(\bar{B}C^2, \bar{B}C, \bar{B}) + Q(B\phiC^2, \bar{B}\phi\bar{C}, \bar{B}\phi)\}.\]

By definition (1.17),

\[(2.8) \quad Q(\bar{B}C^2, \bar{B}C, \bar{B}) = BC(-1)G(B^2C^3)G(B^2C^2)G(B^2C)G(\bar{B}C^2)G(\bar{B}C)G(\bar{B})/G(B^3C^3).\]

Define

\[(2.9) \quad W(B, C) = G(B^2C^3)G(B^2C^2)G(B^2C)G(\bar{B}C^2)G(\bar{B}C)G(\bar{B})G(B^3C^3)/q.\]
By (2.8) and (2.9),

\[(2.10)\quad W(B, C) = Q(BC^2, BC, B), \quad \text{if } B^3C^3 \neq 1.\]

Assume first that

\[(2.11)\quad B^2, B^2C, \text{ and } B^2C^2 \text{ are nontrivial}.\]

By (2.11), if \(B^3C^3 = 1\), then

\[(2.12)\quad W(B, C) = -q^2 \quad \text{and} \quad Q(BC^2, BC, B) = -a^3.\]

Hence (2.10) has the extension

\[(2.13)\quad (q - 1)q^2\delta(B^3C^3) + Q(BC^2, BC, B) = W(B, C).\]

Since \(\delta(B^3C^6) = \delta(B^3C^3) + \delta(\phi B^3C^3)\), the expression in braces in (2.7) equals

\[(2.14)\quad W(B, C) + W(B\phi, C).\]

Again using (2.11), we thus obtain

\[(2.15)\quad T = P(B, C) + P(B\phi, C).\]

By (2.11) and Theorem 1.1, each summand \(d(C^a, B^2, C)\) in (2.3) vanishes. Thus \(L = T\) and the result follows from (2.15) under the assumption (2.11).

Now drop the assumption (2.11). For brevity, set

\[(2.16)\quad R(a, b) = R(C^a, C^b, C),\]

\[(2.17)\quad U(a, b) = L_3(C^a, C^b, C\phi)/R(a, b),\]

\[(2.18)\quad V(a, b) = S_3(C^a, C^b, C)/R(a, b),\]

where \(0 \leq a, b \leq 2\). Observe that \(R(a, b), U(a, b), V(a, b)\) are symmetric in \(a, b\). We proceed to evaluate these functions.

From (1.3),

\[(2.19)\quad R(0, 0) = G^2(C^2)/G(C^4),\]

\[(2.20)\quad R(1, 0) = G(C)G(C^2)/G(C),\]

\[(2.21)\quad R(2, 0) = R(2, 2) = R(2, 1) = -|G(C^2)|^2G(C^3)G(C)/G^2(C),\]

\[(2.22)\quad R(1, 1) = -G(C^3)G^2(C)/G(C).\]

From (1.4),

\[(2.23)\quad V(0, 0) = \begin{cases} \quad q^{-3}, & \text{if } C = 1, \\ \quad q^{-2}, & \text{if } C = \phi, \\ \quad q^{-1}, & \text{if ord } C = 3 \text{ or } 4, \\ \quad 1, & \text{if ord } C > 4, \end{cases}\]

\[(2.24)\quad V(1, 0) = \begin{cases} \quad q^{-3}, & \text{if } C = 1, \\ \quad q^{-1}, & \text{if ord } C = 2 \text{ or } 3, \\ \quad 1, & \text{if ord } C > 3, \end{cases}\]
\begin{align*}
V(2, 0) &= V(2, 2) = V(1, 1) = \begin{cases}
q^{-3}, & \text{if } C = 1, \\
q^{-2}, & \text{if } C = \phi, \\
q^{-1}, & \text{if } \text{ord } C > 2,
\end{cases} \\
V(2, 1) &= \begin{cases}
q^{-3}, & \text{if } C = 1, \\
q^{-1}, & \text{if } C \neq 1.
\end{cases}
\end{align*}

By [5, Theorem 4.1],
\begin{align*}
U(0, 0) &= \begin{cases}
4 - 3q, & \text{if } C = 1, \\
-q^3 + 3q^2 - 5q + 4, & \text{if } C = \phi, \\
q^2 - 3q + 3, & \text{if } \text{ord } C = 3, \\
q^{-1}, & \text{if } \text{ord } C = 4, \\
1, & \text{if } \text{ord } C > 4.
\end{cases}
\end{align*}

We claim that
\begin{align*}
U(1, 0) &= \begin{cases}
4 - 3q, & \text{if } C = 1, \\
2 - q, & \text{if } C = \phi, \\
q^2 - 3q + 3, & \text{if } \text{ord } C = 3, \\
1, & \text{if } \text{ord } C > 3.
\end{cases}
\end{align*}

The cases \( C = 1, C = \phi \) of (2.28) follow from [5, (2.13), (2.14)]. The case where \( \text{ord } C = 3 \) follows from (2.27), since by [5, Lemmas 2.1, 2.2],
\begin{align*}
U(1, 0) &= U(0, 0) \quad \text{if ord } C = 3.
\end{align*}

The last case where \( \text{ord } C > 3 \) follows from (2.24) and Conjecture 1.1 (note that the hypothesis (1.5) of Theorem 1.1 holds with \( A = \overline{C}, B = 1 \)). Next we claim that
\begin{align*}
U(2, 0) &= U(2, 2) = \begin{cases}
4 - 3q, & \text{if } C = 1, \\
-q^3 + 3q^2 - 5q + 4, & \text{if } C = \phi, \\
2 - q, & \text{if } \text{ord } C > 2.
\end{cases}
\end{align*}

The first equality in (2.29) follows from [5, Lemmas 2.1, 2.2]. The cases \( C = 1, C = \phi \) of (2.29) follow from [5, (2.13), (2.14)], while the remaining case follows from Theorem 1.4. The same argument shows that
\begin{align*}
U(1, 1) &= \begin{cases}
4 - 3q, & \text{if } C = 1, \\
(2-q)/q, & \text{if } C = \phi, \\
2 - q, & \text{if } \text{ord } C > 2.
\end{cases}
\end{align*}

Finally, we claim that
\begin{align*}
U(2, 1) &= \begin{cases}
4 - 3q, & \text{if } C = 1, \\
2 - q, & \text{if } C \neq 1.
\end{cases}
\end{align*}

The cases \( C = 1, C = \phi \) of (2.31) follow from [5, (2.13), (2.14)], while the cases where \( C^2 \neq 1 \) follow from (2.30), since
\begin{align*}
U(2, 1) &= U(1, 1) \quad \text{if ord } C > 2
\end{align*}
by [5, Lemmas 2.1, 2.2].
For \(0 \leq a, b \leq 2\), set
\[
(2.33) \quad d(a, b) = \{U(a, b) - V(a, b)\} R(a, b),
\]
so that by (2.2),
\[
(2.34) \quad d(a, b) = d(\overline{C}^a, \overline{C}^b, C).
\]
From (2.19)–(2.31), we obtain the following evaluation of \(d(a, b)\):
\[
(2.35) \quad d(0, 0) \begin{cases} 
-(4-3a-a^{-3}), & \text{if } C = 1, \\
-(-q^3 + 3q^2 - 5q + 4 - q^{-2}), & \text{if } C = \phi, \\
(q^2 - 3q + 3 - q^{-1})G^3(\overline{C})/q, & \text{if } \text{ord}\, C = 3, \\
0, & \text{if } \text{ord}\, C > 3.
\end{cases}
\]
\[
(2.36) \quad d(1, 0) \begin{cases} 
-(4-3q-q^{-3}), & \text{if } C = 1, \\
-(-q^3 + 3q^2 - 5q + 4 - q^{-2}), & \text{if } C = \phi, \\
(q^2 - 3q + 3 - q^{-1})G^3(\overline{C})/q, & \text{if } \text{ord}\, C = 3, \\
0, & \text{if } \text{ord}\, C > 3.
\end{cases}
\]
\[
(2.37) \quad d(2, 0) = d(2, 2) \begin{cases} 
-(4-3q-q^{-3}), & \text{if } C = 1, \\
-(-q^3 + 3q^2 - 5q + 4 - q^{-2}), & \text{if } C = \phi, \\
(-2 - q - q^{-1})G(\overline{C}^3)G^3(\overline{C})/q, & \text{if } \text{ord}\, C > 2;
\end{cases}
\]
\[
(2.38) \quad d(1, 1) \begin{cases} 
-(4-3q-q^{-3}), & \text{if } C = 1, \\
-(-q^3 + 3q^2 - 5q + 4 - q^{-2}), & \text{if } C = \phi, \\
-(2 - q - q^{-1})G(\overline{C}^3)G^3(\overline{C})C(-1)/q, & \text{if } \text{ord}\, C > 2;
\end{cases}
\]
and
\[
(2.39) \quad d(2, 1) \begin{cases} 
-(4-3q-q^{-3}), & \text{if } C = 1, \\
-(-q^3 + 3q^2 - 5q + 4 - q^{-2}), & \text{if } C = \phi, \\
-(-2 - q - q^{-1})G(\overline{C}^3)G^3(\overline{C})/q, & \text{if } \text{ord}\, C > 2.
\end{cases}
\]
We now evaluate \(L\) from (2.3), using (2.7), (2.8), and (2.35)–(2.39), and Theorem 1.2 follows.

3. APPENDIX

Here we give a proof of Theorem 1.3. Let \(H\) denote the left side of (1.16). First suppose that \(DE = 1\). Then
\[
(3.1) \quad H = \frac{1}{q - 1} \sum_A |G(AE)|^2 |G(A\overline{E})|^2 |G(AF)|^2 |G(A\overline{F})|
\]
\[
= \begin{cases} 
M - (q + 1)G(\overline{E}F)\overline{G}(\overline{E}\overline{F}), & \text{if } E^2 = 1, \\
M - qG(\overline{E}F)\overline{G}(\overline{E}\overline{F}) - qG(\overline{F}E)\overline{G}(\overline{E}\overline{F}), & \text{if } E^2 \neq 1,
\end{cases}
\]
where
\[
(3.2) \quad M = \frac{q^2}{q - 1} \sum_A G(AF)\overline{G}(A\overline{F}).
\]
By (1.1),

$$M = \frac{q^2}{q - 1} \sum_{t, u} \sum_{v} A F(t) \overline{A F(u)} \zeta^{T(t-u)} = q^2 (q - 1) \delta (F^2).$$

Using (3.3) in (3.1), we easily deduce (1.16) in the case $DE = 1$.

By symmetry, it remains to prove (1.16) in the case

$$DE \neq 1, \quad DF \neq 1, \quad EF \neq 1.$$ By (1.1),

$$H = \frac{1}{q - 1} \sum_{t, u, v} \sum_{x, y, z \neq 0} A \left( \frac{tuw}{xyz} \right) D(tx) E(uy) F(vz) \zeta^{T(t+u+v-x-y-z)}$$

$$= \frac{1}{q - 1} \sum_{t, u, v} \sum_{x, y, z} A(tuw) D(txy) E(uyz) F(vzx) \zeta^{T(y(t-1)+z(u-1)+x(v-1))},$$

where the last equality results from replacing $t$ by $ty$, $u$ by $uz$, and $v$ by $vx$. By (3.4), it follows that

$$H = \frac{1}{q - 1} G(DE) G(DF) G(EF)$$

$$\times \sum_{t, u, v} A(tuw) \overline{DE} (1 - t) \overline{EF} (1 - u) \overline{DF} (1 - v) D(t) E(u) F(v).$$

Thus,

$$H/G(DE) G(DF) G(EF)) = \sum_{t, v \neq 0} \overline{DE} (1 - t) \overline{EF} (1 - 1/(tv)) \overline{DF} (1 - v) D(t) \overline{E}(tv) F(v)$$

$$= \sum_{t, v \neq 0} \overline{DE} (1 - t/v) \overline{EF} (1 - 1/t) \overline{DF} (1 - v) D(t/v) \overline{E}(t) F(v)$$

$$= \sum_{t, v} EF(v) DF(t) \overline{EF}(t - 1) \overline{DF}(1 - v) \overline{DE}(v - t)$$

$$EF(-1) \sum_{t, v \neq 0} DF \left( \frac{1 + v}{t} \right) \overline{EF} \left( \frac{1 + t}{v} \right) \overline{DE}(v - t)$$

$$= EF(-1) \{ J(\overline{DE}, DE) J(E, \overline{DE}) + J(\overline{DF}, DE) J(E \phi, \overline{DE}) \},$$

where the last equality follows from [2, (28)]. Since $DE \neq 1$, we can apply the formula [8]

$$J(A, B) = G(A) G(\overline{AB}) A(-1) / G(\overline{B}), \quad \text{if } B \neq 1$$

to express all of the Jacobi sums in (3.7) in terms of Gauss sums. Then (1.16) readily follows.

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