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A CHARACTERIZATION OF SUZUKI'S SIMPLE GROUPS

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ABSTRACT. In this short paper we have characterized Suzuki's simple groups $S_z(2^{2m+1})$, $m \geq 1$ using only the set $\pi_e(G)$ of orders of elements in the group G. That is, we have

Theorem 2. *Let G be a finite group. Then $G \simeq S_z(2^{2m+1})$, $m \geq 1$ if and only if $\pi_e(G) = \{2, 4$, all factors of $(2^{2m+1} - 1)$, $(2^{2m+1} - 2^{m+1} + 1)$, and $(2^{2m+1} + 2^{m+1} + 1)\}$.*

Suzuki's simple groups $S_z(2^{2m+1})$, $m \geq 1$ is a family of Zassenhaus groups (Z-groups) of odd degree [9]. In [4] we characterized another family of Zassenhaus groups of odd degree $L_2(2^m)$ using only the set of orders of elements in the group G. That is, let $\pi_e(G)$ denote the set of orders of elements in the group G. Then we have proved the following theorem.

Theorem 1. *Let G be a finite group. Then $G \simeq L_2(2^m)$, $m \geq 2$ if and only if $\pi_e(G) = \{2$, all factors of $(2^m - 1)$ and $2^m + 1)\}$.*

In this short paper, we continue this work and obtain the following theorem.

Theorem 2. *Let G be a finite group. Then $G \simeq S_z(2^{2m+1})$, $m \geq 1$ if and only if $\pi_e(G) = \{2, 4$, all factors of $(2^{2m+1} - 1)$, $(2^{2m+1} - 2^{m+1} + 1)$, and $(2^{2m+1} + 2^{m+1} + 1)\}$.*

Since the simple Z-groups of odd degree consists of $L_2(2^m) (m \geq 2)$ and $S_z(2^{2m+1})$, $(m \geq 1)$, we have

Corollary. *Let G be a finite group and M a simple Z-groups of odd degree. Then $G \simeq M$ if and only if $\pi_e(G) = \pi_e(M)$.*

Before starting the proof we give a remark about the set $\pi_e(G)$ in Theorem 2. Since $2^{2m+1} - 1 \not\equiv 0 \pmod 3$ and $(2^{2m+1} - 2^{m+1} + 1) \cdot (2^{2m+1} + 2^{m+1} + 1) = 2^{4m+2} + 1 \not\equiv 0 \pmod 3$, $3 \notin \pi_e(G)$. And $2^{2m+1} - 1$ is prime to 5, but $5 \in \pi_e(G)$ by $2^{4m+2} + 1 \equiv 0 \pmod 5$.

Proof of Theorem 2. We need only prove the sufficiency by [3, XI Theorem 3.10].

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Let G be a finite group satisfying the condition of the theorem. Then G is a CIT group [10].

1. G is a nonsolvable CIT group. Let r be a prime divisor of $2^{2m+1}-1$. Then we have $r \neq 2$ and $r \neq 5$. Suppose that G is solvable. Then G contains a Hall $\{2, 5, r\}$-subgroup C. The assumptions imply that C is a solvable group in which every element has prime power order. From [2, Theorem 1] we have $|\pi(C)| \leq 2$; a contradiction.

Then G has a normal 2-group N such that G/N is isomorphic to one of the following groups [10, III, Theorem 5]:

> $L_2(q)$, $q = 2^n$, $n \geq 2$ or $q = p$ is a Fermat prime or Mersenne prime, or $q = 9$; $S_z(2^{2n+1})$, $n \geq 1$; $L_3(4)$; M_9.

2. $\overline{G} = G/N \simeq S_z(2^{2m+1})$, $m \geq 1$. Since $3 \notin \pi_e(G)$, $\overline{G}$ can only be $S_z(2^{2n+1})$, $n \leq m$. Again by the assumption, the maximal order of elements of $\overline{G}$ is $2^{2m+1} + 2^{m+1} + 1$. We infer that $n = m$.

3. $G \simeq S_z(2^{2m+1})$, $m \geq 1$. It is enough to prove that $N = 1$. If not, we can assume $N \neq 1$ is an elementary abelian 2-group without loss of generality. Considering the Sylow 5-normalizer $N_{\overline{G}}(\overline{P}_5)$ of $\overline{P}_5$ in $\overline{G}$, where $\overline{P}_5 \in \mathrm{Syl}_5\,\overline{G}$, we see that the cyclic subgroup $\mathbb{Z}_4$ is contained in $N_{\overline{G}}(\overline{P}_5)$. Then $\overline{G}$ has a subgroup $\overline{U} = \mathbb{Z}_4[\mathbb{Z}_5]$. Again considering the inverse image U of $\overline{U}$ in G, $|U| = 2^2 \cdot 5 \cdot |N|$ and U has a normal series

$$U > \mathbb{Z}_5[N] > N > 1.$$

Taking an element g of order 5 in U, there is an element c of order 4 in U such that c normalizes $\langle g\rangle$, i.e. $c^{-1}gc = g^2$. Set $H = \langle g, c\rangle$. Then $H = \mathbb{Z}_4[\mathbb{Z}_5]$ and $U = HN$. Since the elementary abelian 2-group N admits the involutive automorphism induced by the element $t = c^2$, we have $|C_N(t)|^2 \geq |N|$. Suppose $|C_N(t)|^2 > |N|$. As every squares t_1 and t_2 of elements of order 4 in H are conjugate, we consider the centralizer $C_N(t_1)$ and $C_N(t_2)$ of t_1 and t_2 in N, where $t_1 \neq t_2$. Then $C_N(t_1t_2) = C_N(t_1) \cap C_N(t_2) \neq 1$ for every such t_1, t_2. But $|t_1t_2| = 5$, G contains element of order 10; a contradiction. And if $|C_N(t)|^2 = |N|$, we have $C_N(t) = [N, t]$ by $(n \cdot n^t)^t = n \cdot n^t$, $\forall n \in N$. In this case we prove that U has the elements of order 8. If not, then $G^4 = 1$. Put $x \in N$ and $[x, c] \neq 1$, we have

$$(xc)^2 = x^2x^{-1}cxc^{-1}c^2 = [x, c^{-1}]t$$

and

$$(xc)^4 = [[x, c^{-1}], t] = 1.$$

So $[N, c, t] = 1$ and $[N, c] \leq C_N(t) = [N, t]$. Hence $[N, c, c] \leq [N, t, c] = [N, c, t] = 1$ by the three subgroup lemma. Also since $[N, c^2] = [N, c][N, c] \cdot [N, c, c] = 1$, which means that $t = c^2$ centralizes N, it is contrary to $|C_N(t)|^2 = |N|$. Therefore $N = 1$ and $G \simeq S_z(2^{2m+1})$, $m \geq 1$. The theorem is proved.

Remark 1. In [5, Theorem 3] we have characterized $S_z(2^{2m+1})$ using the conditions of the order of a group G and the set of orders of elements in G. Theorem 2 is clearly a generalization of the above-mentioned conclusion.

Remark 2. The condition of odd degree cannot be removed. In fact, we cannot characterize $L_2(9)$, a simple Z-group of even degree, using only the condition of $\pi_e(G)$.

Remark 3. Except $L_2(9) \simeq A_6$, we have characterized all simple groups in which every element has prime power order [11, Theorem 16].

That is, we have the following conclusion:

Let G be a finite group. Then
$G \simeq A_5$ iff $\pi_e(G) = \{1, 2, 3, 5\}$ [8];
$G \simeq L_2(7)$ iff $\pi_e(G) = \{1, 2, 3, 4, 7\}$ [6];
$G \simeq L_2(8)$ iff $\pi_e(G) = \{1, 2, 3, 7, 9\}$ [4];
$G \simeq L_2(17)$ iff $\pi_e(G) = \{1, 2, 3, 4, 8, 9, 17\}$ [1];
$G \simeq L_3(4)$ iff $\pi_e(G) = \{1, 2, 3, 4, 5, 7\}$ [7];
$G \simeq S_z(8)$ iff $\pi_e(G) = \{1, 2, 4, 5, 7, 13\}$;
$G \simeq S_z(32)$ iff $\pi_e(G) = \{1, 2, 4, 5, 25, 31, 41\}$.

Open problem. How many finite simple groups G can be characterized using only the condition of $\pi_e(G)$? I think it is a difficulty problem.

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