

ON MULTIPLE SALIÉ SUMS

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ABSTRACT. This note shows that the Davenport-Hasse relation for Gauss sums is equivalent to the evaluation of some multidimensional exponential sums that generalize that of Salié.

This paper shows that the Davenport-Hasse relation for Gauss sums is equivalent to the evaluation of some multidimensional exponential sums that generalize that of Salié. In addition to being interesting in their own right, these sums also occur in the analysis of Fourier expansions of certain Poincaré series on the n -fold cover of $GL(n)$. For example, if one considers the analogues of the exponential sums in Theorem 1 of M. Larsen's appendix to [B-F-G] that occur for the Poincaré series on the 3-fold cover of $GL(3)$, the hyper-Kloosterman sums that arise are of the type considered here when $n = 3$. Thus the evaluation given below may turn out to be important in the further development of the theory of such automorphic forms when $n \geq 3$. This is certainly the case for $n = 2$, as the work of Iwaniec [I] shows.

Let $F = GF(q)$, $q = p^r$ for p prime, and $n \in \mathbf{Z}^+$ be such that $q \equiv 1 \pmod{n}$. For ψ a multiplicative character of order n and $a \in F^*$ consider the n -dimensional Salié sum defined by

$$S_n(a) = \sum_{\substack{x_1 x_2 \cdots x_n = a \\ x_i \in F^*}} \psi(x_2 x_3^2 \cdots x_n^{n-1}) e_q(x_1 + \cdots + x_n),$$

where $e_q(x) = e(\text{tr } x/p)$. (Note that $S_n(a)$ does not depend on the choice of ψ .)

Theorem.

$$(*) \quad S_n(a) = \varepsilon_{q,n} q^{(n-1)/2} \sum_{x : x^n = a} e_q(nx),$$

where

$$\varepsilon_{q,n} = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ (-1)^{(q-1)(n-2)/8 + r-1} i^{(p-1)^2 r/4} & \text{if } n \text{ is even.} \end{cases}$$

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In fact (*) is equivalent to the Davenport-Hasse relation, which states that for any multiplicative character χ

$$(**) \quad \tau(\chi)\tau(\chi\psi)\cdots\tau(\chi\psi^{n-1}) = \tau(\chi^n)\bar{\chi}^n(n)\tau(\psi)\tau(\psi^2)\cdots\tau(\psi^{n-1})$$

where $\tau(\chi) = \sum_{x \in F^*} \chi(x)e_q(x)$. For χ with $\chi^n = 1$ this formula is trivial, and otherwise it was obtained by Davenport and Hasse in [D-H] (see also [L, p. 61]). In case $n = 3$ and $q = p$ a proof of (**) that parallels one for the triplication formula for the Γ -function has been given by Greene and Stanton [G-S].

To prove (*) first multiply both sides of (**) by $\bar{\chi}(a)$ and sum over χ , i.e., take the Fourier transform. This yields

$$S_n(a) = \tau(\psi)\tau(\psi^2)\cdots\tau(\psi^{n-1}) \sum_{x : x^n=a} e_q(nx).$$

Now (*) follows from the classical formulas

$$\begin{aligned} \tau(\psi^k)\tau(\psi^{n-k}) &= \psi^k(-1)q, & k = 1, \dots, n-1, \\ \tau(\varphi) &= (-1)^{r-1}i^{(p-1)^2r/4}q^{1/2}, & \text{for } p > 2, \end{aligned}$$

where φ is of order 2, together with the fact that $\psi(-1) = -1$ iff n is even and $(q-1)/n$ is odd (see for example [L-N, Theorem 5.15 and Remark 5.13]). The converse is similar.

In the case of $n = 2$ this yields Salié's original result [S] (Salié worked with $q = p$)

$$\sum_x \varphi(x) e_q(x + a\bar{x}) = \varepsilon_{q,2}\sqrt{q} \sum_{x : x^2=a} e_q(2x),$$

while for $n = 3$ it gives

$$\sum_{x_1x_2x_3=a} \psi(x_2x_3^2) e_q(x_1 + x_2 + x_3) = q \sum_{x : x^3=a} e_q(3x).$$

We remark that many other multiple exponential sums can be similarly treated, for example those studied by Mordell in [M]. Also, $S_3(a^3)$ was independently evaluated by N. Elkies (unpublished) by an elegant argument using elliptic curves.

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