AN EXTENSION OF A THEOREM OF KHAVINSON

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Abstract. We extend a generalization of the Cauchy-Green formula, which in turn extends results in rational approximation.

Theorem. Let $K \subseteq \mathbb{C}$ be compact and have finite perimeter. Let $2 < p < \infty$ and $f \in W^{1,p}(\mathbb{C})$. Then there exists a subset $E$ of $K$ satisfying $m_2E = 0$ and

$$\frac{1}{2\pi i} \int_{B_k} \frac{f(z)}{z - z_0} \, dz - \frac{1}{\pi} \int_K \frac{\partial f}{\partial z} \, dA = \begin{cases} f(z_0) & \text{if } z_0 \in K \setminus E, \\ 0 & \text{if } z_0 \in \mathbb{C} \setminus K. \end{cases}$$

Remarks. (1) $B_K$ is the reduced boundary of $K$, the "nice" portion of the topological boundary of $K$. For a precise definition of $B_K$ and of finite perimeter, see [2]. $W^{1,p}(\mathbb{C})$ is defined in [1]. $dm_2 = dA$ denotes Lebesgue area measure and $\partial$ indicates $\partial/\partial z$.

(2) Khavinson [2, Theorem 2.1] proves the above for $f \in \text{Lip}(1, \mathbb{C})$. The above theorem does extend Khavinson's; this will be discussed following the proof.

Proof. By Theorem 3.18 [1], there exists a sequence $(\phi_n)$ of $C^\infty$—functions having compact support, with $\phi_n \to f$ in $W^{1,p}(\mathbb{C})$. By Theorem 5.4 [1], $W^{1,p}(\mathbb{C})$ is continuously embedded in $(C \cap L^\infty)(\mathbb{C})$, the bounded continuous functions (with the sup norm).

Set $F = \left\{ z_0 \in K : \int_{B_k} \frac{|dz|}{|z - z_0|} < \infty \text{ and for each } n, \phi_n(z_0) = \frac{1}{2\pi i} \int_{B_k} \frac{\phi_n}{z - z_0} \, dz - \frac{1}{\pi} \int_K \frac{\partial \phi_n}{\partial z} \, dA \right\}$, and $E = K \setminus F$. By Theorem 2.1 [2], $m_2E = 0$.

Let $z_0 \in \mathbb{C} \setminus E$. Since $(\phi_n)$ converges uniformly to $f$, we have

$$\left| \int_{B_k} \frac{f - \phi_n}{z - z_0} \, dz \right| \leq \|f - \phi_n\|_{B_k} \int_{B_k} \frac{|dz|}{|z - z_0|} \to 0,$$

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where \( \| \cdot \|_{B_K} \) denotes the supremum over the set \( B_K \). Writing \( 1/p + 1/q = 1 \), we see
\[
\left| \int_K \frac{\partial (f - \phi_n)}{z - z_0} \, dA \right| \leq \| \partial (f - \phi_n) \|_{L^p(K)} \left( \int_K \frac{dA}{|z - z_0|^q} \right)^{1/q} \to 0.
\]

Since \( \phi_n(z_0) \to f(z_0) \), we are done. (By Theorem 2.1 [2], the above theorem holds for each \( \phi_n \).) \( \Box \)

Consider now the above theorem together with Theorem 2.1 [2]. The conclusions of both theorems involve only the values of the function in a neighborhood of \( K \). By means of a cutoff function (multiply by a \( C^\infty \)—function having compact support, which is equal to 1 in a neighborhood of \( K \)), we can (in the hypotheses of the above theorems) replace the spaces \( W^{1,p}(\mathbb{C}) \) and \( \text{Lip}_c(1, \mathbb{C}) \) by \( W^{1,p}_c(\mathbb{C}) \) and \( \text{Lip}_c(1, \mathbb{C}) \), respectively, where the subscript \( c \) refers to compact support. In the proof of Theorem 2.1 [2], Khavinson (basically) proves the inclusion \( \text{Lip}_c(1, \mathbb{C}) \subseteq W^{1,p}_c(\mathbb{C}) \). Clearly \( W^{1,\infty}_c(\mathbb{C}) \subseteq W^{1,p}_c(\mathbb{C}) \) for \( 2 < p < \infty \).

It is in this manner that the above theorem is an extension of Khavinson's.

Using the above theorem, we may extend Theorem 3.1, Theorem 3.2, and Corollary 3.1 of [2] to functions in \( W^{1,p}(\mathbb{C}) \). The statements and proofs are otherwise identical: simply replace \( f \in \text{Lip}(1, \mathbb{C}) \) by \( f \in W^{1,p}(\mathbb{C}) \).

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**References**


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