PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 114, Number 3, MARCH 1992

AN EXTENSION OF A THEOREM OF KHAVINSON

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(Communicated by Paul S. Muhly)

ABSTRACT. We extend a generalization of the Cauchy-Green formula, which in turn extends results in rational approximation.

Theorem. *Let $K \subseteq \mathbb{C}$ be compact and have finite perimeter. Let $2 < p < \infty$ and $f \in W^{1,p}(\mathbb{C})$. Then there exists a subset E of K satisfying $m_2 E = 0$ and*

$$\frac{1}{2\pi i}\int_{B_K}\frac{f}{z-z_0}\,dz - \frac{1}{\pi}\iint_K \frac{\bar\partial f}{z-z_0}\,dA = \begin{cases} f(z_0) & \text{if } z_0 \in K \setminus E, \\ 0 & \text{if } z_0 \in \mathbb{C}\setminus K. \end{cases}$$

Remarks. (1) B_K is the reduced boundary of K, the "nice" portion of the topological boundary of K. For a precise definition of B_K and of finite perimeter, see [2]. $W^{1,p}(\mathbb{C})$ is defined in [1]. $dm_2 = dA$ denotes Lebesgue area measure and $\bar\partial$ indicates $\partial/\partial\bar z$.

(2) Khavinson [2, Theorem 2.1] proves the above for $f \in \mathrm{Lip}\,(1, \mathbb{C})$. The above theorem does extend Khavinson's; this will be discussed following the proof.

Proof. By Theorem 3.18 [1], there exists a sequence (ϕ_n) of C^∞—functions having compact support, with $\phi_n \to f$ in $W^{1,p}(\mathbb{C})$. By Theorem 5.4 [1], $W^{1,p}(\mathbb{C})$ is continuously embedded in $(C \cap L^\infty)(\mathbb{C})$, the bounded continuous functions (with the sup norm).

$$\text{Set } F = \left\{ z_0 \in K : \int_{B_K}\frac{d|z|}{|z-z_0|} < \infty \text{ and for each } n, \right.$$
$$\left. \phi_n(z_0) = \frac{1}{2\pi i}\int_{B_K}\frac{\phi_n}{z-z_0}\,dz - \frac{1}{\pi}\iint_K\frac{\bar\partial\phi_n}{z-z_0}\,dA \right\},$$

and $E = K \setminus F$. By Theorem 2.1 [2], $m_2 E = 0$.

Let $z_0 \in \mathbb{C} \setminus E$. Since (ϕ_n) converges uniformly to f, we have

$$\left|\int_{B_K}\frac{f-\phi_n}{z-z_0}\,dz\right| \le \|f-\phi_n\|_{B_K}\int_{B_K}\frac{d|z|}{|z-z_0|} \longrightarrow 0,$$

Received by the editors September 19, 1990.
1991 *Mathematics Subject Classification.* Primary 30E10, 31A10.
Key words and phrases. Cauchy-Green, rational approximation, finite perimeter, reduced boundary, Sobolev spaces, Lipschitz class.

where $\|\cdot\|_{B_K}$ denotes the supremum over the set B_K. Writing $1/p+1/q=1$, we see

$$\left|\iint_K \frac{\bar{\partial}(f-\phi_n)}{z-z_0}\,dA\right| \le \|\bar{\partial}(f-\phi_n)\|_{L^p(K)}\left(\iint_K \frac{dA}{|z-z_0|^q}\right)^{1/q} \longrightarrow 0.$$

Since $\phi_n(z_0)\to f(z_0)$, we are done. (By Theorem 2.1 [2], the above theorem holds for each ϕ_n.) □

Consider now the above theorem together with Theorem 2.1 [2]. The conclusions of both theorems involve only the values of the function in a neighborhood of K. By means of a cutoff function (multiply by a C^∞—function having compact support, which is equal to 1 in a neighborhood of K), we can (in the hypotheses of the above theorems) replace the spaces $W^{1,p}(\mathbb{C})$ and Lip $(1, \mathbb{C})$ by $W_c^{1,p}(\mathbb{C})$ and $\mathrm{Lip}_c(1, \mathbb{C})$, respectively, where the subscript c refers to compact support. In the proof of Theorem 2.1 [2], Khavinson (basically) proves the inclusion $\mathrm{Lip}_c(1, \mathbb{C}) \subseteq W_c^{1,\infty}(\mathbb{C})$. Clearly $W_c^{1,\infty}(\mathbb{C}) \subsetneqq W_c^{1,p}(\mathbb{C})$ for $2<p<\infty$.

It is in this manner that the above theorem is an extension of Khavinson's.

Using the above theorem, we may extend Theorem 3.1, Theorem 3.2, and Corollary 3.1 of [2] to functions in $W^{1,p}(\mathbb{C})$. The statements and proofs are otherwise identical: simply replace $f\in \mathrm{Lip}(1, \mathbb{C})$ by $f\in W^{1,p}(\mathbb{C})$.

Acknowledgment

I wish to thank Xiao Li for finding a mistake in my original proof of the theorem and for bringing the paper [2] to my attention

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