ABSTRACT. We show that Schwartz distributions have kernels in the class of the pointwise nonstandard functions.

The main purpose of this note is to show that every Schwartz distribution $T \in \mathcal{D}'$ has a kernel $f: *\mathbb{R}^d \to *\mathbb{C}$ in the class of the pointwise nonstandard functions in the sense that

$$\langle T, \varphi \rangle = \int_{\mathbb{R}^d} f(x) \varphi(x) \, dx$$

for all $\varphi \in \mathcal{D}$, where $*\varphi$ is the nonstandard extension of $\varphi$. Recall that, in general, the Schwartz distributions do not have kernels in the class of the standard pointwise functions (Schwartz [2]).

We denote the usual classes of the $C^\infty$-functions, $C^\infty$-functions with compact supports, and continuous complex-valued functions defined on $\mathbb{R}^d$ ($d$ is a natural number) by $\mathcal{E} \equiv \mathcal{E}(\mathbb{R}^d) \equiv C^\infty(\mathbb{R}^d)$, $\mathcal{D} \equiv \mathcal{D}(\mathbb{R}^d) \equiv C^\infty_0(\mathbb{R}^d)$, and $C^0 \equiv C^0(\mathbb{R}^d) \equiv C(\mathbb{R}^d)$, respectively, and the class of Schwartz distributions by $\mathcal{D}' \equiv \mathcal{D}'(\mathbb{R}^d)$. Let $\mathcal{P}$ be the ring of standard complex-valued polynomials defined on $\mathbb{R}^d$. As usual, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ will be the systems of the natural, real, and complex numbers, respectively, and we use also the notations $N_0 = \{0\} \cup \mathbb{N}$ and $f(x) = f(-x)$.

In what follows, we shall work in a nonstandard model with a set of individuals $J$ that contains the complex numbers $\mathbb{C}$ and degree of saturation $k$ larger than $2^\kappa$ for $\kappa = \text{card } C^0$. In particular, any polysaturated model of $\mathbb{C}$ will suffice (Stroyan and Luxemburg [3]). If $X$ is a set of complex numbers or a set of (standard) functions, then $*X$ will be its nonstandard extension, and if $f: X \to Y$ is a (standard) mapping, then $*f: *X \to *Y$ will be its nonstandard extension. We shall use the same notation, $*$, for the convolution operator $*: \mathcal{D}' \times \mathcal{D} \to \mathcal{E}$ and for its nonstandard extension $*: *\mathcal{D}' \times *\mathcal{D} \to *\mathcal{E}$.

Lemma. There exists $\Delta$ in $*\mathcal{D}$ such that for all $\varphi$ in $C^0$ we have

$$\int_{*\mathbb{R}^d} \Delta(x) \varphi(x) \, dx = \int_{*\mathbb{R}^d} \tilde{\Delta}(x) \varphi(x) \, dx = \varphi(0).$$

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For the proof we refer the reader to [4].

**Proposition.** If $T$ is a Schwartz distribution, then (1) holds for $f = T * \Delta$ and all $\phi$ in $\mathcal{D}$.

**Proof.** Using the properties of the convolution operator, the transfer principle, and the above lemma (since $T * \phi$ is in $\mathcal{E}$), we obtain
\[
\int_{\mathbb{R}^d} (T * \Delta)(x) \phi(x) \, dx = (T * \Delta, \phi) = (T, \phi * \Delta)
\]
\[
= (T * \phi)(0) = (T, \phi)
\]
as required. □

We shall keep $\Delta$ fixed in what follows.

**Corollary.** (i) The mapping from $\mathcal{D}'$ into $\mathcal{E}$ defined by $T \rightarrow T * \Delta$ is injective and preserves the addition, multiplication by a complex (standard) number, and partial differentiation in $\mathcal{D}'$.

(ii) There exists an infinitely large natural number $\nu \in \mathcal{N}$ such that $P * \Delta = P$ holds for all (nonstandard, in general) polynomials $P \in \mathcal{P}$ with degree not higher than $\nu$. In particular, $P = P * \Delta = *P$ holds for all standard polynomials $P \in \mathcal{P}$.

(iii) If $f$ is a continuous function, then $f * \Delta$ is an extension of $f$.

**Proof.** (i) By the transfer principle, $T * \Delta \in \mathcal{E}$ for all $T$ in $\mathcal{D}'$, while $T * \Delta = 0$ (in $\mathcal{E}$) implies $T = 0$ (in $\mathcal{D}'$), by the above proposition. The preservation of linear operations follows immediately from the corresponding property of the convolution operator and transfer principle.

(ii) Define the internal set
\[
\Omega_\Delta = \left\{ n \in \mathcal{N} : \int_{\mathbb{R}^d} \Delta(x) x^\alpha \, dx = 0, \quad 1 \leq |\alpha| \leq n, \quad \alpha \in \mathcal{N}_0^d \right\}
\]
and observe that, by our lemma, $\mathcal{N} \subseteq \Omega_\Delta$. Hence, by overflow, $\Omega_\Delta$ contains an infinitely large number $\nu$. Suppose now, that $\xi \in \mathcal{R}^d$. The hyperfinite ($\ast$-finite) Taylor's expansion of $P$ at $\xi$ gives
\[
(\Delta * P)(\xi) = \int_{\mathbb{R}^d} \Delta(x) P(\xi - x) \, dx
\]
\[
= P(\xi) + \sum_{|\alpha| = 1}^\nu \frac{(-1)^{|\alpha|}}{\alpha!} \int_{\mathbb{R}^d} \Delta(x) x^\alpha \, dx = P(\xi),
\]
as required. The equality $*P * \Delta = *P$ follows as a particular case since the degree of a standard polynomial is always finite and hence less than $\nu$.

(iii) follows immediately from our lemma for $\phi = f_x$ and standard $x \in \mathbb{R}^d$, where $f_x(\xi) = f(x - \xi)$. The proof is complete. □

**Remark** (Multiplication of distributions). Consider $\mathcal{E}$ as a differential algebra over $\mathcal{C}$ with respect to pointwise addition, multiplication, and internal partial differentiation. Notice now that the space of Schwartz distributions $\mathcal{D}'$ is isomorphically embedded in $\mathcal{E}$ through the above injection and hence the
Schwartz distributions can be multiplied within an associative algebra (something impossible in $\mathcal{D}'$ itself). Further, the operations in $^{*}\mathcal{E}$ generalize the usual operations with polynomials in the sense the $\mathcal{P}$ (considered as a subset of $\mathcal{D}'$) is a differential subalgebra of $^{*}\mathcal{E}$ over $\mathbb{C}$. The multiplication in $^{*}\mathcal{E}$ also generalizes the usual multiplication in $C^{0}$ (considered as a subset of $\mathcal{D}'$) although in a somewhat weaker sense: if $f$ and $g$ are two continuous functions and $^{*}f*\Delta$ and $^{*}g*\Delta$ are their images in $^{*}\mathcal{E}$, then their product $(^{*}f*\Delta)(^{*}g*\Delta)$ in $^{*}\mathcal{E}$ is an extension of the usual product $fg$ in $C^{0}$. We wish to pay attention to the similarity between the class of nonstandard functions $^{*}\mathcal{E}$ (in the context discussed above) and the classes of generalized functions introduced (in the framework of standard analysis) by Colombeau [1] with the same purpose: multiplication of Schwartz distributions.

REFERENCES