UNIFORM LIMITS OF SEQUENCES OF POLYNOMIALS AND THEIR DERIVATIVES

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(Communicated by Palle E. T. Jorgensen)

Abstract. Let $E$ be a compact subset of the unit interval $[0, 1]$, and let $C(E)$ denote the space of functions continuous on $E$ with the uniform norm. Consider the densely defined operator $D: C(E) \to C(E)$ given by $Dp = p'$ for all polynomials $p$. Let $\mathcal{G}$ represent the graph of $D$, that is $\mathcal{G} = \{(p, p'): p$ polynomials$\}$ considered as a submanifold of $C(E) \times C(E)$. Write the interior of the set $E$, $\text{int} E$ as a countable union of disjoint open intervals and let $\bar{E}$ be the union of the closure of these intervals. The main result is that the closure of $\mathcal{G}$ is equal to the set of all functions $(h, k) \in C(E) \times C(E)$ such that $h$ is absolutely continuous on $\bar{E}$ and $k|\bar{E} = h'|\bar{E}$. As a consequence, the operator $D$ is closable if and only if the set $E$ is the closure of its interior. On the other extreme, $\mathcal{G}$ is dense in $C(E) \times C(E)$ i.e. for any pair $(f, g) \in C(E) \times C(E)$, there exists a sequence of polynomials $(p_n)$ so that $p_n \to f$ and $p'_n \to g$ uniformly on $E$, if and only if the interior $\text{int} E$ of $E$ is empty.

1. Introduction

The goal of this paper is to characterize those pairs of continuous functions $(f, g)$ which arise as uniform limits of polynomial pairs of the form $(p_n, p'_n)$ (where $p'_n$ is the derivative of the polynomial $p_n$) on some compact set $E \subset \mathbb{R}$. In the language of operator theory, this result amounts to a characterization of the uniform closure $\mathcal{G}^\perp$ of the graph $\mathcal{G}$ of the differentiation operator $D: C(E) \to C(E)$ given by $Dp = p'$ with domain equal to the dense linear manifold of all polynomials in $C(E)$. An explicit description of the uniform closure of $\mathcal{G}$ is given by (2.1) and Theorems 2.2 and 2.9. As a corollary, a complete characterization of when $D$ is closable (i.e., when $\mathcal{G}^\perp$ contains no nontrivial elements of the form $(0, k)$) is obtained in Theorem 3.1; for a discussion of this problem in the plane, see [FR]. On the other extreme, another specialization (Corollary 3.3) gives a necessary and sufficient condition for the graph of $\mathcal{G}$ to be dense in $C(E) \times C(E)$, i.e., for an arbitrary pair $(f, g)$ in $C(E) \times C(E)$ to be uniformly approximable over $E$ by a sequence of the form $(p_n, p'_n)$, with $p_n$ a polynomial. Partial results on this latter problem for the case where $E$ is a compact subset of the complex plane are obtained in [B1, B2]. The general problem for the complex case where $E$ is a compact subset...
of $C$ appears to be quite difficult; our purpose here is to show that a complete, explicit solution can be obtained for the case that $E$ is contained in $\mathbb{R}$ by use of elementary functional analysis.

More complicated parallel versions of these results where $C(E) \times C(E)$ is replaced by $L^p(\mu) \times L^p(\nu)$ where $\mu$ and $\nu$ are compactly supported measures on $\mathbb{R}$, together with connections with some operator theory questions, are presented in [BF]. Most of these results (including the results of this paper) form part of the second author's Ph.D. dissertation [F] written under the direction of the first author.

2. The closure of the graph of a differentiation

Let $E$ be a compact subset of the real line $\mathbb{R}$, and let $C(E)$ denote the Banach space of continuous functions on $E$ with the uniform norm. For convenience without loss of generality we assume that $E$ is contained in the unit interval $[0, 1]$. Consider the densely defined operator $D: C(E) \to C(E)$ given by $Dp = p'$ for all polynomials $p$. Let $\mathcal{G}$ represent the graph of $D$, that is

$$\mathcal{G} = \{(p, p') : p \text{ a polynomial}\}.$$ 

We wish to characterize the closure of $\mathcal{G}$, $\mathcal{G}^-$ in the product topology of uniform convergence in each component.

The analysis relies on the topological properties of $E$. Denote by $\text{int} E$ the interior of $E$. Write $\text{int} E$ as a countable disjoint union of intervals $\text{int} E = \bigcup_i (a_i, b_i)$. Let $\hat{E}$ be given by $\hat{E} = \bigcup_i [a_i, b_i]$. Notice first that the expression for $\hat{E}$ is also a disjoint union; indeed, if $[a_i, b_i] \cap [a_j, b_j] \neq \emptyset$ where $a_i < a_j$, then $(a_i, b_j) \subset \text{int} E$, a contradiction. Secondly $E \setminus \hat{E}$ contains no intervals since it is a subset of the nowhere dense set $E \setminus \text{(int} E)$. Now let $\hat{\mathcal{G}}$ denote the closed submanifold of $C(E) \times C(E)$ defined by

$$(2.1) \quad \hat{\mathcal{G}} = \{(h, k) \in C(E) \times C(E) : h \text{ is } AC(\hat{E}) \text{ and } k|\hat{E} = h'|\hat{E}\}.$$ 

Our result is that $\mathcal{G}^- = \hat{\mathcal{G}}$. We begin by proving $\mathcal{G}^- \subset \hat{\mathcal{G}}$, which is straightforward. The other inclusion relies on a characterization of the annihilator of $\mathcal{G}$, $\mathcal{G}^\perp$, where the duality relationship is that of $C(E)$ with $M(E)$, the Banach space of all regular Borel measures supported in $E$.

**Theorem 2.2.** $\mathcal{G}^- \subset \hat{\mathcal{G}}$.

**Proof.** Let $(h, k) \in \mathcal{G}^-$. Let $\{p_n\}$ be a sequence of polynomials such that $p_n \to h$ and $p'_n \to k$ uniformly on $E$. If $\hat{E} = \bigcup_i [a_i, b_i]$, we need to show for each $i$, $h$ is $AC([a_i, b_i])$ and $h'|[a_i, b_i] = k$. Fix one such $i$. We have, for all $x, y \in [a_i, b_i]$,

$$\begin{align*}
p_n(x) - p_n(y) &= \int_{[y,x]} p'_n(t) \, dt.
\end{align*}$$

Furthermore for each $x, y \in [a_i, b_i]$, $p_n(x) - p_n(y) \to h(x) - h(y)$.

Finally, since $p'_n \to k$ uniformly, we have

$$\lim_{n \to \infty} \int_{[y,x]} p'_n(t) \, dt = \int_{[y,x]} k(t) \, dt.$$
for all $x, y \in [a_i, b_i]$. The theorem then follows by taking the limit as $n$ goes to infinity in (2.2.1). \qed

The proof of the other inclusion involves establishing some lemmas which characterize $\mathcal{G}^\perp$ and localizes our concerns to each $[a_i, b_i]$.

**Lemma 2.3.** For two regular Borel measures $m_1$ and $m_2$ supported on $E$, $(m_1, m_2) \in \mathcal{G}^\perp$ if and only if

\begin{align*}
(2.3.1) & \quad \int_{[0,1]} dm_1 = 0 \\
(2.3.2) & \quad \text{the measure } \left[ \int_{[x,1]} m_1(t) \right] dx \text{ is supported in } E, \\
(2.3.3) & \quad dm_2(x) = -\left[ \int_{[x,1]} m_1(t) \right] dx.
\end{align*}

**Proof.** We first note that (2.3.3) states that the Radon-Nikodym derivative of $m_2$ with respect to $m$ (Lebesgue measure) is $-\left[ \int_{[x,1]} m_1(t) \right]$. That is

$$dm_2/dm = -\left[ \int_{[x,1]} m_1(t) \right].$$

Now suppose $(m_1, m_2) \in \mathcal{G}^\perp$. Then for all polynomials $p$,

\begin{align*}
0 &= \int_{[0,1]} p(x) dm_1(x) + \int_{[0,1]} p'(x) dm_2(x) \\
&= p(0) \int_{[0,1]} dm_1(x) + \int_{[0,1]} p'(t) dt \int_{[x,1]} dm_1(t) + \int_{[0,1]} p'(x) dm_2(x) \\
&= p(0) \int_{[0,1]} dm_1(t) + \int_{[0,1]} p'(x) \int_{[x,1]} dm_1(t) dx + \int_{[0,1]} p'(x) dm_2(x).
\end{align*}

Letting $p = 1$ in (2.3.4), we get that $m_1$ must satisfy (2.3.1). Next, by considering polynomials $p'$ with $p(0) = 0$ (a dense set in $C([0,1])$ by the Stone-Weierstrass theorem), it follows from (2.3.4) that $m_1$ and $m_2$ must satisfy (2.3.2) and (2.3.3).

To prove the other direction simply reverse the argument. \qed

The key to the proof of the general result is that elements in $\mathcal{G}^\perp$ must be zero off $\hat{E}$. This is established in the following.

**Lemma 2.4.** If $(m_1, m_2) \in \mathcal{G}^\perp$, then $m_1$ has no point masses in $E \setminus \hat{E}$.

**Proof.** As noted above, $E \setminus \hat{E}$ contains no intervals. Thus any open interval must intersect the complement of $E \setminus \hat{E}$. Let $x_0 \in E \setminus \hat{E}$. By (2.3.2), for $m$ — a.e. $x \in [0, 1] \setminus E$

\begin{equation}
(2.4.1) \quad m_1((x, 1]) = \int_{[x,1]} dm_1(t) = 0.
\end{equation}
Thus, by additivity, \( m_1((x,y]) = 0 \) for \( m \)-a.e. \( x, y \) in \([0,1]\setminus E\). Now choose a sequence of positive numbers \( \{\xi_n\} \) so that \( \xi_n \searrow 0 \). Let \( I_n = (x_0 - \xi_n, x_0 + \xi_n) \). We wish to show that, for each \( n \), there are points \( x_n \) and \( y_n \) belonging to \( I_n \), with \( x_n < x_0 < y_n \), so that \( m_1((x_n, y_n]) = 0 \). To do this, from (2.4.1), it suffices to show that each \( I_n \) intersects the complement of \( E \) in a set of positive Lebesgue measure on each side of \( x_0 \). We first show that \( I_n \) must intersect the complement of \( E \) on each side of \( x_0 \). Indeed, if, say, \( I_n \) did not intersect \([0,1]\setminus E\) to the left of \( x_0 \), then it would follow that \( (x_0 - \xi_n, x_0] \subset E \). But this would contradict \( x_0 \notin \tilde{E} \). Next, note that both \([0,1]\setminus E\) and \( I_n \) are open. Thus their nonempty intersection, which, from above, contains an open interval on each side of \( x_0 \), must have positive Lebesgue measure. Thus we have shown that there are sequences \( \{x_n\}, \{y_n\} \) such that \( \bigcap_n (x_n, y_n] = \{x_0\} \) and \( m_1((x_n, y_n]) = 0 \) for each \( n \). Finally, since \( m_1 \) is regular, we have

\[
\lim_{n \to \infty} m_1((x_n, y_n]) = m_1(\{x_0\})
\]

and the lemma follows. \( \Box \)

**Lemma 2.5.** Let \( A = \{x \in E\setminus \tilde{E}: \int_{[x,1]} m_1(t) \, dt = 0\} \). Then \( m_1(A) = 0 \).

**Proof.** We use in our arguments the total variation of \( m_1 \), \( |m_1| \), which is regular if \( m_1 \) is regular. So, without loss of generality, we may assume \( A \) is closed. Let \( \epsilon > 0 \) be given and choose an open set \( U \supset A \) so that \( |m_1(U \setminus A)| < \epsilon \). Write \( U \) as a disjoint union of open intervals: \( U = \bigcup_n (\alpha_n, \beta_n) \). We construct a new open set \( U_1 \supset A \) as follows. Delete each interval \( (\alpha_n, \beta_n) \) for which \( (\alpha_n, \beta_n) \cap A = \emptyset \) from the collection \( \{(\alpha_n, \beta_n)\} \). Let

\[
\alpha'_n = \inf\{x: x \in (\alpha_n, \beta_n) \cap A\}, \quad \beta'_n = \sup\{x: x \in (\alpha_n, \beta_n) \cap A\}.
\]

Then, for each \( n \) not deleted, let

\[
U_1 = \bigcup_n (\alpha'_n, \beta'_n).
\]

Then \( A \subset U_1 \), modulo the endpoints \( \alpha'_n \) and \( \beta'_n \) which are in \( A \), since \( A \) is closed. But the collection of all endpoints is a countable set in \( E\setminus \tilde{E} \), and, by Lemma 2.4, has \( m_1 \)-measure zero. Furthermore,

\[
m_1((\alpha'_n, \beta'_n)) = \int_{(\alpha'_n, 1]} m_1(t) \, dt - \int_{(\beta'_n, 1]} m_1(t) \, dt = 0
\]

by assumption. Thus \( m_1(U_1) = 0 \). Finally, then

\[
|m_1(A)| = |m_1(U_1) - m_1(A)| = |m_1(U_1 \setminus A)|
\]

\[
\leq |m_1(U \setminus A)| \leq |m_1(U \setminus A)| < \epsilon.
\]

So by arbitrariness of \( \epsilon \), Lemma 2.5 follows. \( \Box \)

**Lemma 2.6.** For \((m_1, m_2) \in \mathcal{F} \perp, |m_1|(E\setminus \tilde{E}) = 0 \).

**Proof.** Suppose there is an \( A \subset E\setminus \tilde{E} \) with \( m_1(A) \neq 0 \). Then by Lemma 2.4, there exists \( x_0 \in E\setminus \tilde{E} \) so that

\[
\int_{[x_0, 1]} m_1(t) \, dt \neq 0.
\]
But, since \( m_1 \) has no point masses in \( E \setminus \hat{E} \), the function \( \int_{(x, 1]} dm_1(x) \) is continuous on \( E \setminus \hat{E} \).

Thus there is an interval \( I \) containing \( x_0 \) and \( \xi > 0 \), so that
\[
\left| \int_{(x, 1]} dm_1(t) \right| > \xi
\]
for all \( x \in I \cap (E \setminus \hat{E}) \). Since \( I \) must intersect \([0, 1] \setminus E\) and \( m_1 \) is supported on \( E \), this contradicts (2.3.2). \( \square \)

**Corollary 2.7.** For \((m_1, m_2) \in \mathcal{G}^\perp, \quad |m_2|(E \setminus \hat{E}) = 0.\)

**Proof.** Let \( B \subset E \setminus \hat{E} \). Then, if \( A \) is defined as in Lemma 2.5,
\[
B = (B \cap A) \cup (B \cap [(E \setminus \hat{E}) \setminus A]) = B_1 \cup B_2.
\]
Now, by (2.3.3), for any set \( C \)
\[
\int_C dm_2(t) = -\int_C \int_{(x, 1]} dm_1(t) dx.
\]
So \( m_2(B_1) = 0 \) by definition of \( A \). So suppose \( m_2(B_2) \neq 0 \). Then there must be an \( x_0 \in E \setminus \hat{E} \) so that \( \int_{(x_0, 1]} dm_1(t) \neq 0 \). But this leads to a contradiction as in the proof of Lemma 2.6. \( \square \)

Recall that \([a_i, b_i] \quad (i = 1, 2, \ldots)\) are the maximal intervals in \( \hat{E} \).

**Lemma 2.8.** For each \( i \),
\[
0 = \int_{[a_i, 1]} dm_1(t) = \int_{(b_i, 1]} dm_1(t).
\]

**Proof.** By regularity of \( m \), for \( x_n \not\in a_i \).
\[
(2.8.1) \quad \lim_{n \to \infty} m_1((x_n, 1]) = m_1([a_i, 1]).
\]
Let \( I \) be an open interval containing \( a_i \). Then \( I \) must intersect \([0, 1] \setminus E\), an open set of positive Lebesgue measure. But since (2.3.2) states \( \int_{(x, 1]} dm(t) = 0 \) for \( m - \text{a.e.} \ x \in [0, 1] \setminus E \), there is a sequence \( \{x_n\} \) converging up to \( a_i \) so that, for each \( n \),
\[
m_1((x_n, 1]) = \int_{(x_n, 1]} dm_1(t) = 0.
\]
Using this sequence in (2.8.1), we get \( \int_{[a_i, 1]} dm(t) = 0 \). A similar argument shows that \( \int_{(b_i, 1]} dm_1(t) = 0 \). \( \square \)

**Theorem 2.9.** \( \mathcal{G} \subseteq \mathcal{G}^- \).

**Proof.** Let \((h, k) \in \mathcal{G}^\perp \). We show \((h, k) \in \mathcal{G}^- \) by showing
\[
\int_{[0, 1]} h(x) dm_1(x) + \int_{[0, 1]} k(x) dm_2(x) = 0
\]
for all \((m_1, m_2) \in \mathcal{G}^\perp \). But, using Lemma 2.6 and Corollary 2.7, it suffices to show for each \( i \) and for each \((m_1, m_2) \in \mathcal{G}^\perp \)
\[
\int_{[a_i, b_i]} h(x) dm_1(x) + \int_{[a_i, b_i]} k(x) dm_2(x) = 0.
\]
So fix \( i \), and let \((m_1, m_2) \in \mathcal{F}^\perp\). Using (2.3.3) we have
\[
\int_{[a, b]} k(x) \, dm_2(x) = -\int_{[a, b]} k(x) \left( \int_{[x, 1]} \, dm_1(t) \right) \, dx
\]
\[
= -\int_{[a, b]} \left( \int_{[a, x]} k(t) \, dt \right) \, dm_1(x)
\]
\[
- \int_{[b, 1]} \left( \int_{[a, b]} k(t) \, dt \right) \, dm_1(x)
\]
by interchanging the order of integration. But, by assumption, \( h(y) - h(x) = \int_{(x, y]} k(t) \, dt \) for all \( x, y \in [a, b] \). Thus,
\[
\int_{[a, b]} k(x) \, dm_2(x) = -\int_{[a, b]} h(x) \, dm_1(x) + h(a_i) \int_{[a, b]} \, dm_1(x)
\]
\[
- h(b_i) \int_{(b, 1]} dm_1(x) + h(a_i) \int_{[a, 1]} \, dm_1(x)
\]
\[
= -\int_{[a, b]} h(x) \, dm_1(x) + h(a_i) \int_{[a, 1]} \, dm_1(x)
\]
\[
- h(b_i) \int_{(b, 1]} dm_1(x).
\]
So
\[
\int_{[a, b]} h(x) \, dm_1(x) + \int_{[a, b]} k(x) \, dm_2(x)
\]
\[
= h(a_i) \int_{[a, 1]} \, dm_1(x) - h(b_i) \int_{(b, 1]} \, dm_1(x) = 0
\]
by Lemma 2.8. \( \square \)

3. THE CLOSABILITY OF DIFFERENTIATION
AND SIMULTANEOUS APPROXIMATIONS

We conclude by describing when \( D \) is closable and when \( \mathcal{F}^- \) is all of \( C(E) \). The latter provides the solution to the approximation problem initially addressed.

**Theorem 3.1.** \( D \) is closable if and only if \( E = (\hat{E})^- \) (that is, \( E \) is the closure of its interior).

**Proof.** First suppose \( (\hat{E})^- = E \). Let \((0 \oplus g) \in \mathcal{F}^-\). Then, by Theorem 2.2, \( g \equiv 0 \) on \( \hat{E} \). Thus, by continuity, \( g \equiv 0 \) on \( (\hat{E})^- \).

Conversely, suppose \( (\hat{E})^- \neq E \). Let \( x_0 \in E \setminus (\hat{E})^- \). By Urysohn's Lemma, there is a function \( g \), continuous on \( E \), so that \( g(x_0) = 1 \) and \( g(x) = 0 \) for all \( x \in (\hat{E})^- \). Now \((0, g) \in \mathcal{F} \), by Theorem 2.9, but is nonzero. Thus \( D \) is not closable. \( \square \)

**Theorem 3.2.** \( \mathcal{F}^- = C(E) \times C(E) \) if and only if \( \hat{E} = \emptyset \) (that is, \( E \) has empty interior).

**Proof.** If \( \hat{E} \neq \emptyset \), we have for \( h(x) \equiv 1 \) on \( E \), \((h \oplus h) \) is in \( C(E) \oplus C(E) \) but not in \( \mathcal{F}^- \). Conversely, if \( \hat{E} = \emptyset \), then there are no requirements on an element of \( C(E) \times C(E) \) to be in \( \mathcal{F} \). Thus \( \mathcal{F}^- = C(E) \times C(E) \). \( \square \)
A reformulation of Theorem 3.2 is as follows.

**Corollary 3.3.** If $E$ is a compact subset of the real line and $f$ and $g$ are arbitrary continuous on $E$, then there is a sequence of polynomials $\{p_n\}$ so that $p_n \to f$ and $p_n' \to g$ uniformly on $E$ if and only if $E$ has empty interior.

**REFERENCES**


