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UNIFORM LIMITS OF SEQUENCES OF POLYNOMIALS AND THEIR DERIVATIVES

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ABSTRACT. Let E be a compact subset of the unit interval $[0, 1]$, and let $C(E)$ denote the space of functions continuous on E with the uniform norm. Consider the densely defined operator $D: C(E) \to C(E)$ given by $Dp = p'$ for all polynomials p. Let $\mathscr{G}$ represent the graph of D, that is $\mathscr{G} = \{(p, p'): p$ polynomials$\}$ considered as a submanifold of $C(E) \times C(E)$. Write the interior of the set E, $\operatorname{int} E$ as a countable union of disjoint open intervals and let $\widehat{E}$ be the union of the closure of these intervals. The main result is that the closure of $\mathscr{G}$ is equal to the set of all functions $(h, k) \in C(E) \times C(E)$ such that h is absolutely continuous on $\widehat{E}$ and $k|\widehat{E} = h'|\widehat{E}$. As a consequence, the operator D is closable if and only if the set E is the closure of its interior. On the other extreme, $\mathscr{G}$ is dense in $C(E) \times C(E)$ i.e. for any pair $(f, g) \in C(E) \times C(E)$, there exists a sequence of polynomials $\{p_n\}$ so that $p_n \to f$ and $p_n' \to g$ uniformly on E, if and only if the interior $\operatorname{int} E$ of E is empty.

1. INTRODUCTION

The goal of this paper is to characterize those pairs of continuous functions (f, g) which arise as uniform limits of polynomial pairs of the form (p_n, p_n') (where p_n' is the derivative of the polynomial p_n) on some compact set $E \subset \mathbb{R}$. In the language of operator theory, this result amounts to a characterization of the uniform closure $\mathscr{G}^-$ of the graph $\mathscr{G}$ of the differentiation operator $D: C(E) \to C(E)$ given by $Dp = p'$ with domain equal to the dense linear manifold of all polynomials in $C(E)$. An explicit description of the uniform closure of $\mathscr{G}$ is given by (2.1) and Theorems 2.2 and 2.9. As a corollary, a complete characterization of when D is closable (i.e., when $\mathscr{G}^-$ contains no nontrivial elements of the form $(0, k)$) is obtained in Theorem 3.1; for a discussion of this problem in the plane, see [FR]. On the other extreme, another specialization (Corollary 3.3) gives a necessary and sufficient condition for the graph of $\mathscr{G}$ to be dense in $C(E) \times C(E)$, i.e., for an arbitrary pair (f, g) in $C(E) \times C(E)$ to be uniformly approximable over E by a sequence of the form (p_n, p_n'), with p_n a polynomial. Partial results on this latter problem for the case where E is a compact subset of the complex plane are obtained in [B1, B2]. The general problem for the complex case where E is a compact subset

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of $\mathbb{C}$ appears to be quite difficult; our purpose here is to show that a complete, explicit solution can be obtained for the case that E is contained in $\mathbb{R}$ by use of elementary functional analysis.

More complicated parallel versions of these results where $C(E) \times C(E)$ is replaced by $L^p(\mu) \times L^p(\nu)$ where μ and ν are compactly supported measures on $\mathbb{R}$, together with connections with some operator theory questions, are presented in [BF]. Most of these results (including the results of this paper) form part of the second author's Ph.D. dissertation [F] written under the direction of the first author.

2. The closure of the graph of a differentiation

Let E be a compact subset of the real line $\mathbb{R}$, and let $C(E)$ denote the Banach space of continuous functions on E with the uniform norm. For convenience without loss of generality we assume that E is contained in the unit interval $[0, 1]$. Consider the densely defined operator $D\colon C(E) \to C(E)$ given by $Dp = p'$ for all polynomials p. Let $\mathscr{G}$ represent the graph of D, that is

$$\mathscr{G} = \{(p, p')\colon p \text{ a polynomial}\}.$$

We wish to characterize the closure of $\mathscr{G}$, $\mathscr{G}^-$ in the product topology of uniform convergence in each component.

The analysis relies on the topological properties of E. Denote by $\operatorname{int} E$ the interior of E. Write $\operatorname{int} E$ as a countable disjoint union of intervals $\operatorname{int} E = \dot{\bigcup}_i (a_i, b_i)$. Let $\widehat{E}$ be given by $\widehat{E} = \dot{\bigcup}_i [a_i, b_i]$. Notice first that the expression for $\widehat{E}$ is also a disjoint union; indeed, if $[a_i, b_i] \cap [a_j, b_j] \neq \varnothing$ where $a_i < a_j$, then $(a_i, b_j) \subset \operatorname{int} E$, a contradiction. Secondly $E \backslash \widehat{E}$ contains no intervals since it is a subset of the nowhere dense set $E \backslash (\operatorname{int} E)$. Now let $\widehat{\mathscr{G}}$ denote the closed submanifold of $C(E) \times C(E)$ defined by

$$\widehat{\mathscr{G}} = \{(h, k) \in C(E) \times C(E)\colon h \text{ is } AC(\widehat{E}) \text{ and } k|\widehat{E} = h'|\widehat{E}\}. \tag{2.1}$$

Our result is that $\mathscr{G}^- = \widehat{\mathscr{G}}$. We begin by proving $\mathscr{G}^- \subset \widehat{\mathscr{G}}$, which is straightforward. The other inclusion relies on a characterization of the annihilator of $\mathscr{G}$, $\mathscr{G}^\perp$, where the duality relationship is that of $C(E)$ with $M(E)$, the Banach space of all regular Borel measures supported in E.

Theorem 2.2. $\mathscr{G}^- \subseteq \widehat{\mathscr{G}}$.

Proof. Let $(h, k) \in \mathscr{G}^-$. Let $\{p_n\}$ be a sequence of polynomials such that $p_n \to h$ and $p_n' \to k$ uniformly on E. If $\widehat{E} = \dot{\bigcup}_i [a_i, b_i]$, we need to show for each i, h is $AC([a_i, b_i])$ and $h'|_{[a_i, b_i]} = k$. Fix one such i. We have, for all $x, y \in [a_i, b_i]$,

$$p_n(x) - p_n(y) = \int_{[y, x]} p_n'(t)\, dt. \tag{2.2.1}$$

Furthermore for each $x, y \in [a_i, b_i]$, $p_n(x) - p_n(y) \to h(x) - h(y)$.

Finally, since $p_n' \to k$ uniformly, we have

$$\lim_{n \to \infty} \int_{[y, x]} p_n'(t)\, dt = \int_{[y, x]} k(t)\, dt$$

for all $x, y \in [a_i, b_i]$. The theorem then follows by taking the limit as n goes to infinity in (2.2.1). □

The proof of the other inclusion involves establishing some lemmas which characterize $\mathscr{G}^\perp$ and localizes our concerns to each $[a_i, b_i]$.

Lemma 2.3. *For two regular Borel measures m_1 and m_2 supported on E, $(m_1, m_2) \in \mathscr{G}^\perp$ if and only if*

$$\int_{[0,1]} dm_1 = 0 \tag{2.3.1}$$

$$\text{the measure } \left[\int_{(x,1]} dm_1(t)\right] dx \text{ is supported in } E, \tag{2.3.2}$$

and

$$dm_2(x) = -\left[\int_{(x,1]} dm_1(t)\right] dx. \tag{2.3.3}$$

Proof. We first note that (2.3.3) states that the Radon-Nikodym derivative of m_2 with respect to m (Lebesgue measure) is $-[\int_{(x,1]} dm_1(t)]$. That is

$$dm_2/dm = -\left[\int_{(x,1]} dm(t)\right].$$

Now suppose $(m_1, m_2) \in \mathscr{G}^\perp$. Then for all polynomials p,

(2.3.4)

$$\begin{aligned} 0 &= \int_{[0,1]} p(x)\, dm_1(x) + \int_{[0,1]} p'(x)\, dm_2(x) \\ &= p(0) \int_{[0,1]} dm_1(x) + \int_{[0,1]} \left(\int_{[0,x]} p'(t)\, dt\right) dm_1(x) + \int_{[0,1]} p'(x)\, dm_2(x) \\ &= p(0) \int_{[0,1]} dm_1(t) + \int_{[0,1]} p'(x) \left(\int_{(x,1]} dm_1(t)\right) dx + \int_{[0,1]} p'(x)\, dm_2(x). \end{aligned}$$

Letting $p \equiv 1$ in (2.3.4), we get that m_1 must satisfy (2.3.1). Next, by considering polynomials p' with $p(0) = 0$ (a dense set in $C([0, 1])$ by the Stone-Weierstrass theorem), it follows from (2.3.4) that m_1 and m_2 must satisfy (2.3.2) and (2.3.3).

To prove the other direction simply reverse the argument. □

The key to the proof of the general result is that elements in $\mathscr{G}^\perp$ must be zero off $\widehat{E}$. This is established in the following.

Lemma 2.4. *If $(m_1, m_2) \in \mathscr{G}^\perp$, then m_1 has no point masses in $E \backslash \widehat{E}$.*

Proof. As noted above, $E \backslash \widehat{E}$ contains no intervals. Thus any open interval must intersect the complement of $E \backslash \widehat{E}$. Let $x_0 \in E \backslash \widehat{E}$. By (2.3.2), for m – a.e. $x \in [0, 1] \backslash E$

$$m_1((x, 1]) = \int_{(x,1]} dm_1(t) = 0. \tag{2.4.1}$$

Thus, by additivity, $m_1((x, y]) = 0$ for m – a.e. x, y in $[0, 1]\backslash E$. Now choose a sequence of positive numbers $\{\xi_n\}$ so that $\xi_n \searrow 0$. Let $I_n = (x_0 - \xi_n, x_0 + \xi_n)$. We wish to show that, for each n, there are points x_n and y_n belonging to I_n, with $x_n < x_0 < y_n$, so that $m_1((x_n, y_n]) = 0$. To do this, from (2.4.1), it suffices to show that each I_n intersects the complement of E in a set of positive Lebesgue measure on each side of x_0. We first show that I_n must intersect the complement of E on each side of x_0. Indeed, if, say, I_n did not intersect $[0, 1]\backslash E$ to the left of x_0, then it would follow that $(x_0 - \xi_n, x_0] \subset E$. But this would contradict $x_0 \notin \widehat{E}$. Next, note that both $[0, 1]\backslash E$ and I_n are open. Thus their nonempty intersection, which, from above, contains an open interval on each side of x_0, must have positive Lebesgue measure. Thus we have shown that there are sequences $\{x_n\}$, $\{y_n\}$ such that $\bigcap_n (x_n, y_n] = \{x_0\}$ and $m_1((x_n, y_n]) = 0$ for each n. Finally, since m_1 is regular, we have

$$\lim_{n\to\infty} m_1((x_n, y_n]) = m_1(\{x_0\})$$

and the lemma follows. □

Lemma 2.5. *Let* $A = \{x \in E\backslash\widehat{E}\colon \int_{(x,1]} dm_1(t) = 0\}$. *Then* $m_1(A) = 0$.

Proof. We use in our arguments the total variation of m_1, $|m_1|$, which is regular if m_1 is regular. So, without loss of generality, we may assume A is closed. Let $\varepsilon > 0$ be given and choose an open set $U \supset A$ so that $|m_1|(U\backslash A) < \varepsilon$. Write U as a disjoint union of open intervals: $U = \dot{\bigcup}_n (\alpha_n, \beta_n)$. We construct a new open set $U_1 \supset A$ as follows. Delete each interval (α_n, β_n) for which $(\alpha_n, \beta_n) \cap A = \varnothing$ from the collection $\{(\alpha_n, \beta_n)\}$. Let

$$\alpha'_n = \inf\{x\colon x \in (\alpha_n, \beta_n) \cap A\}, \qquad \beta'_n = \sup\{x\colon x \in (\alpha_n, \beta_n) \cap A\}.$$

Then, for each n not deleted, let

$$U_1 = \bigcup_n (\alpha'_n, \beta'_n).$$

Then $A \subset U_1$, modulo the endpoints α'_n and β'_n which are in A, since A is closed. But the collection of all endpoints is a countable set in $E\backslash\widehat{E}$, and, by Lemma 2.4, has m_1-measure zero. Furthermore,

$$m_1((\alpha'_n, \beta'_n)) = \int_{(\alpha'_n, 1]} dm_1(t) - \int_{(\beta'_n, 1]} dm_1(t) = 0$$

by assumption. Thus $m_1(U_1) = 0$. Finally, then

$$\begin{aligned} |m_1(A)| &= |m_1(U_1) - m_1(A)| = |m_1(U_1\backslash A)| \\ &\le |m_1|(U_1\backslash A) \le |m_1|(U\backslash A) < \varepsilon. \end{aligned}$$

So by arbitrariness of ε, Lemma 2.5 follows. □

Lemma 2.6. *For* $(m_1, m_2) \in \mathscr{G}^\perp$, $|m_1|(E\backslash\widehat{E}) = 0$.

Proof. Suppose there is an $A \subset E\backslash\widehat{E}$ with $m_1(A) \neq 0$. Then by Lemma 2.4, there exists $x_0 \in E\backslash\widehat{E}$ so that

$$\int_{(x_0, 1]} dm_1(t) \neq 0.$$

But, since m_1 has no point masses in $E\backslash\widehat{E}$, the function $\int_{(x,1]} dm_1(x)$ is continuous on $E\backslash\widehat{E}$.

Thus there is an interval I containing x_0 and $\xi > 0$, so that

$$\left|\int_{(x,1]} dm_1(t)\right| > \xi$$

for all $x \in I \cap (E\backslash\widehat{E})$. Since I must intersect $[0,1]\backslash E$ and m_1 is supported on E, this contradicts (2.3.2). □

Corollary 2.7. *For* $(m_1, m_2) \in \mathscr{G}^{\perp}$, $|m_2|(E\backslash\widehat{E}) = 0$.

Proof. Let $B \subset E\backslash\widehat{E}$. Then, if A is defined as in Lemma 2.5,

$$B = (B \cap A) \dot{\cup} (B \cap [(E\backslash\widehat{E})\backslash A]) = B_1 \dot{\cup} B_2.$$

Now, by (2.3.3), for any set C

$$m_2(C) = \int_C dm_2(t) = -\int_C \int_{(x,1]} dm_1(t)\,dx.$$

So $m_2(B_1) = 0$ by definition of A. So suppose $m_2(B_2) \neq 0$. Then there must be an $x_0 \in E\backslash\widehat{E}$ so that $\int_{(x_0,1]} dm_1(t) \neq 0$. But this leads to a contradiction as in the proof of Lemma 2.6. □

Recall that $[a_i, b_i]$ $(i = 1, 2, \dots)$ are the maximal intervals in $\widehat{E}$.

Lemma 2.8. *For each* i,

$$0 = \int_{[a_i,1]} dm_1(t) = \int_{(b_i,1]} dm_1(t).$$

Proof. By regularity of m, for $x_n \nearrow a_i$.

$$\lim_{n\to\infty} m_1((x_n, 1]) = m_1([a_i, 1]). \tag{2.8.1}$$

Let I be an open interval containing a_i. Then I must intersect $[0,1]\backslash E$, an open set of positive Lebesgue measure. But since (2.3.2) states $\int_{(x,1]} dm(t) = 0$ for m – a.e. $x \in [0,1]\backslash E$, there is a sequence $\{x_n\}$ converging up to a_i so that, for each n,

$$m_1((x_n, 1]) = \int_{(x_n,1]} dm_1(t) = 0.$$

Using this sequence in (2.8.1), we get $\int_{[a,1]} dm(t) = 0$. A similar argument shows that $\int_{(b_i,1]} dm_1(t) = 0$. □

Theorem 2.9. $\widehat{\mathscr{G}} \subseteq \mathscr{G}^{-}$.

Proof. Let $(h, k) \in \widehat{\mathscr{G}}$. We show $(h, k) \in \mathscr{G}^{-}$ by showing

$$\int_{[0,1]} h(x)\,dm_1(x) + \int_{[0,1]} k(x)\,dm_2(x) = 0$$

for all $(m_1, m_2) \in \mathscr{G}^{\perp}$. But, using Lemma 2.6 and Corollary 2.7, it suffices to show for each i and for each $(m_1, m_2) \in \mathscr{G}^{\perp}$

$$\int_{[a_i,b_i]} h(x)\,dm_1(x) + \int_{[a_i,b_i]} k(x)\,dm_2(x) = 0.$$

So fix i, and let $(m_1, m_2) \in \mathscr{G}^{\perp}$. Using (2.3.3) we have

$$\begin{aligned}\int_{[a_i, b_i]} k(x)\, dm_2(x) &= -\int_{[a_i, b_i]} k(x) \left(\int_{(x, 1]} dm_1(t) \right) dx \\ &= -\int_{[a_i, b_i]} \left(\int_{[a_i, x]} k(t)\, dt \right) dm_1(x) \\ &\quad - \int_{(b_i, 1]} \left(\int_{[a_i, b_i]} k(t)\, dt \right) dm_1(x)\end{aligned}$$

by interchanging the order of integration. But, by assumption, $h(y) - h(x) = \int_{[x, y]} k(t)\, dt$ for all $x, y \in [a_i, b_i]$. Thus,

$$\begin{aligned}\int_{[a_i, b_i]} k(x)\, dm_2(x) &= -\int_{[a_i, b_i]} h(x)\, dm_1(x) + h(a_i) \int_{[a_i, b_i]} dm_1(x) \\ &\quad - h(b_i) \int_{(b_i, 1]} dm_1(x) + h(a_i) \int_{(b_i, 1]} dm_1(x) \\ &= -\int_{[a_i, b_i]} h(x)\, dm_1(x) + h(a_i) \int_{[a_i, 1]} dm_1(x) \\ &\quad - h(b_i) \int_{(b_i, 1]} dm_1(x).\end{aligned}$$

So

$$\begin{aligned}&\int_{[a_i, b_i]} h(x)\, dm_1(x) + \int_{[a_i, b_i]} k(x)\, dm_2(x) \\ &\quad = h(a_i) \int_{[a_i, 1]} dm_1(x) - h(b_i) \int_{(b_i, 1]} dm_1(x) = 0\end{aligned}$$

by Lemma 2.8. □

3. The closability of differentiation and simultaneous approximations

We conclude by describing when D is closable and when $\mathscr{G}^-$ is all of $C(E) \times C(E)$. The latter provides the solution to the approximation problem initially addressed.

Theorem 3.1. *D is closable if and only if $E = (\widehat{E})^-$ (that is, E is the closure of its interior).*

Proof. First suppose $(\widehat{E})^- = E$. Let $(0 \oplus g) \in \mathscr{G}^-$. Then, by Theorem 2.2, $g \equiv 0$ on $\widehat{E}$. Thus, by continuity, $g \equiv 0$ on $(\widehat{E})^-$.

Conversely, suppose $(\widehat{E})^- \neq E$. Let $x_0 \in E \backslash (\widehat{E})^-$. By Urysohn's Lemma, there is a function g, continuous on E, so that $g(x_0) = 1$ and $g(x) = 0$ for all $x \in (\widehat{E})^-$. Now $(0, g) \in \widehat{\mathscr{G}} \subset \mathscr{G}^-$, by Theorem 2.9, but is nonzero. Thus D is not closable. □

Theorem 3.2. *$\mathscr{G}^- = C(E) \times C(E)$ if and only if $\widehat{E} = \varnothing$ (that is, E has empty interior).*

Proof. If $\widehat{E} \neq \varnothing$, we have for $h(x) \equiv 1$ on E, $(h \oplus h)$ is in $C(E) \oplus C(E)$ but not in $\mathscr{G}^-$. Conversely, if $\widehat{E} = \varnothing$, then there are no requirements on an element of $C(E) \times C(E)$ to be in $\widehat{\mathscr{G}} \subset \mathscr{G}^-$. Thus $\mathscr{G}^- = C(E) \times C(E)$. □

A reformulation of Theorem 3.2 is as follows.

Corollary 3.3. *If E is a compact subset of the real line and f and g are arbitrary continuous on E, then there is a sequence of polynomials $\{p_n\}$ so that $p_n \to f$ and $p_n' \to g$ uniformly on E if and only if E has empty interior.*

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