A REMARK ON SINGULAR INTEGRALS WITH COMPLEX HOMOGENEITY

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Abstract. The Hilbert transform can be approximated by operators with Fourier multiplier given by \( c_\gamma \text{sgn}(\xi) |\xi|^{\gamma} \). If we let \( \gamma \to 0 \) it is known that these operators converge to the Hilbert transform in \( L^2 \) norm. It has also been shown that as \( \gamma \to 0 \) these operators may diverge even for functions in \( L^2 \). In this paper it is shown that we also will have divergence if we select any sequence \( \{\gamma_n\}_{n=1}^{\infty} \) such that \( \gamma_n \to 0 \). The proof makes use of Bourgain's entropy method.

Introduction

Let \( f \) be an integrable function on the real line \( \mathbb{R} \). The Hilbert transform \( \hat{f} \) is the principal value convolution of \( f \) with the singular kernel \( \frac{1}{\pi t} \). At almost every point, the integral defining \( \hat{f} \) converges, and hence \( \hat{f} \) exists. The existence of \( \hat{f} \) is a central fact in the theory of real and complex analysis, and one which has been generalized in a variety of directions. In the 1950s a possible new approach to establishing the existence of this important operator was suggested. In one dimension the idea was to replace convolution with \( \frac{1}{\pi t} \) by convolution with \( \frac{1}{\pi t} |t|^{-i\gamma} \). The hope was that this family of operators would be easier to study because of the additional cancellation induced by the term \( |t|^{-i\gamma} \), and that by letting \( \gamma \to 0 \), the properties of the Hilbert transform could be recovered. The first step in this program was successfully completed by Muckenhoupt [5] who showed that for each fixed \( \gamma \),

\[
H_\gamma f = f * \frac{1}{\pi} \frac{\text{sgn} t}{|t|^{1 + i\gamma}}
\]

exists, and that \( H_\gamma f \to H_0 f = \hat{f} \) in \( L^2 \) norm. Later P. Ash [2] showed that \( H_\gamma f \to H_0 f = \hat{f} \) in \( L^p \) norm, \( 1 < p < \infty \). Unfortunately in 1979 it was shown that \( \lim_{\gamma \to 0^+} H_\gamma f(x) \) may fail to exist a.e. In [1] this is shown by construction of a function \( f \in L^p(\mathbb{R}) \), for all \( p, \ 1 < p < \infty \), such that

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lim sup_{γ→0} H_{γ}f(x) = ∞ a.e. This example made use of the fact that the limit was being taken through a continuous parameter. In particular, for each fixed x, there was a set of γ's (of measure zero) such that H_{γ}f(x) = ∞. This left open the question of what happens if we use only a sequence \{γ_j\}_{j=1}^{∞} such that γ_j → 0. In this paper we show that in fact for any sequence \{γ_j\}_{j=1}^{∞} such that lim_{j→∞} γ_j = 0, there exists a function f ∈ L^p(ℝ), 1 < p < ∞, such that on a set of positive measure lim_{j→∞} H_{γ_j}f(x) fails to converge.

The operators given by convolution with \frac{1}{2π}(\text{sgn } t/|t|^{1+γ}) have Fourier multipliers given by c_γ \text{sgn}(ξ)|ξ|^{iγ} where c_γ = 2Γ(-iγ) \sinh(\frac{πγ}{2}) is a constant that depends on γ and converges to a finite nonzero limit as γ → 0. (See [2, 5].)

**The periodic case**

We will first consider the easier case of ℝ replaced by T = [0, 2π). On T we consider the operators given by the multipliers c_γ \text{sgn}(n)|n|^{iγ}. The a.e. existence of these operators has been shown by Weiss and Zygmund [8]. These operators will be denoted by T_{γ}, thus if \varphi(x) = ∑_n \bar{φ}(n)e^{inx} then T_{γ}\varphi(x) = ∑_n \bar{φ}(n)c_γ \text{sgn}(n)|n|^{iγ}e^{inx}.

**Theorem 1.** Let \{γ_j\}_{j=1}^{∞} be a sequence such that lim_{j→∞} γ_j = 0; then there exists a function φ ∈ L^∞(T) such that lim_{j→∞} T_{γ_j}\varphi(x) fails to exist on a set of positive measure.

The proof will use Bourgain's Entropy method. The form of his theorem that will be most useful here is stated below. (See [3, 6] as well as Bourgain's original paper [4] for additional applications of this useful result.)

**Bourgain's Entropy Theorem** [4, Proposition 2]. Let (Ω, μ) denote a probability space. Assume \{T_n\}_{n=1}^{∞} is a sequence of uniformly bounded linear operators on L^2(Ω) and that these operators commute with a sequence \{R_j\} of positive isometries that have R_j(1) = 1 and satisfy \frac{1}{2} ∑_j |R_j f|^2 → \int f^2 dμ in mean for f ∈ L^1(Ω). Under these conditions, if there is a number β > 0 such that for each N ≥ 1 there is a function φ (which is allowed to depend on N), ∥φ∥_2 ≤ 1, and a collection \{T_{n_j}\}_{j=1}^{N} such that ∥T_n φ - T_{n_j} φ∥_2 > β for each i ≠ j, i = 1, ..., N, j = 1, ..., N, then there is bounded function φ such that \{T_n φ\} is not a.e. convergent.

In our application we will let Ω = [0, 2π), θ be the normalized Lebesgue measure, T_n = T_{γ_n}/c_γ and R_j(x) = x + jα mod 2π, where α can be any real number not rationally related to 2π. Since we know that lim_{n→∞} c_γ n exists and is nonzero, if we can show that lim_{n→∞} T_n fails to exist, we can conclude that lim_{n→∞} T_{γ_n} fails to exist.

The required function φ will have the form

φ(x) = \frac{1}{\sqrt{L}} ∑_{n=1}^{L} e^{i2nα}x,

where L will be determined later. Note that ∥φ∥_2 = 1. For this function, we have

T_{γ}\varphi(x) = \frac{1}{\sqrt{L}} ∑_{n=1}^{L} c_γ (\text{sgn } 2^n e^{i2nα}x \log 2^e e^{i2nα}x).
Consequently,

\[ T_r \phi(x) - T_s \phi(x) = \frac{1}{\sqrt{L}} \sum_{n=1}^{L} (\text{sgn } 2^n) e^{i2^n x} e^{i\gamma_n \cdot \ln 2} (1 - e^{i(\gamma_n - \gamma_s) \cdot \ln 2}), \]

and thus

\[ \|T_r \phi - T_s \phi\|_2^2 = \frac{1}{L} \sum_{n=1}^{L} |1 - e^{i(\gamma_n - \gamma_s) \cdot \ln 2}|^2 \]

\[ = \frac{1}{L} \sum_{n=1}^{L} 2(1 - \cos[(\gamma_n - \gamma_s) \cdot \ln 2]). \]

Let \( \theta_{sr} = \frac{1}{\ln 2} (\gamma_n - \gamma_s) \ln 2 \), and define \( \Omega = \{ \theta_{sr} : 1 \leq r < s \leq N \} \). We need to find an \( L \) such that for each \( \theta \in \Omega \), we have \( \frac{1}{L} \sum_{n=1}^{L} 2(1 - \cos 2\pi n \theta) > \beta \). Let \( \Omega = \Omega_1 \cup \Omega_2 \) where

\[ \Omega_1 = \{ \theta : \theta \in \Omega, \theta \text{ irrational} \} \quad \text{and} \quad \Omega_2 = \{ \theta : \theta \in \Omega, \theta \text{ rational} \}. \]

There are two cases. If \( \theta \in \Omega_1 \), we know that \( \{n \theta\} \) is uniformly distributed mod 1, and consequently, since \( (1 - \cos 2\pi x) \) is Riemann integrable and \( \int_0^1 2(1 - \cos 2\pi x) dx = 2 \), there exists an \( l = l(\theta) \) such that

\[ \frac{1}{L} \sum_{n=1}^{L} 2(1 - \cos 2\pi n \theta) > \left( \frac{1}{2} \right)^2 = 1 \]

for all \( L > 1 \). There are only a finite number of numbers in \( \Omega_1 \), we let \( L_1 = \max_{\theta \in \Omega_1} l(\theta) \).

If \( \theta \in \Omega_2 \) then \( \theta \) is rational; we need to prove a similar inequality. Assume \( \theta = p/q \), where \( p \) and \( q \) are integers and \( (p, q) = 1 \). Then we have \( \{n \theta\} = \{np/q\} \). If \( L \) is a multiple of \( q \), then the points \( \{np/q\}_{n=1}^{L} \) are uniformly distributed in the set \( \{k/q\}_{k=0}^{q-1} \). Thus at least \( 1/4 \) of the points will be a distance of at least \( 1/4 \) away from zero. Since \( 1 - \cos(2\pi x) \geq 1 \) for \( 1/4 \leq x \leq 3/4 \), we have

\[ \frac{1}{L} \sum_{n=1}^{L} 2(1 - \cos 2\pi n \theta) \geq \frac{1}{L} \left( \frac{L}{4} \right) 2(1) = \frac{1}{2}. \]

Now let \( L_2 \) be the product \( q_1 q_2 \cdots q_j \) where \( \{p_i/q_i : i = 1, \ldots, j\} \) represent the rationals in \( \Omega_2 \). Finally let \( L = L_1 L_2 \). We see that conditions are satisfied for both estimates above, and hence we have \( \|T_r \phi - T_s \phi\|_2^2 \geq 1/2 \) for \( r = 1, 2, \ldots, N, \ s = 1, 2, \ldots, N, \ r \neq s \) as desired. \( \square \)

**Corollary.** For any \( p, \ 1 < p \leq 2, \) and any \( B > 0, \) there exists a function \( \phi = \phi_B \) in \( L^p(T) \) such that

\[ \left| \left\{ x : \sup_n |\mathcal{T}_{\gamma_n} \phi(x)| > \lambda \right\} \right| > \frac{B}{\lambda^p} ||\phi||_p^p \]

for some \( \lambda > 0 \). In addition, \( \phi \) can be taken to be a trigonometric polynomial.

**Proof.** By Theorem 1, we know that we do not have convergence for some \( \phi \in L^\infty(T) \) and hence for some \( \phi \in L^p(T) \). By Stein's theorem [7], this
implies the failure of a weak type \((p, p)\) inequality for \(1 < p \leq 2\). Because trigonometric polynomials are dense, if we had a weak type inequality for all trigonometric polynomials we would have it for all \(L^p(T)\).

**The Noncompact Case**

We also can state a version of Theorem 1 for the real line.

**Theorem 2.** Let \(\{\gamma_j\}_{j=1}^\infty\) be a sequence such that \(\lim_{n \to \infty} \gamma_n = 0\), then there exists a function \(f \in L^p(\mathbb{R})\), \(1 < p \leq \infty\), such that \(\lim_{n \to \infty} H_{\gamma_n} f(x)\) fails to exist on a set of positive measure.

**Proof.** Bourgain’s Entropy Theorem requires operators to act on a probability space. Thus we cannot use his method to directly give a counterexample on \(\mathbb{R}\). However we will take advantage of our example on \(T\). See [1] where a similar idea is used. Some steps in the computation below are given in greater detail in [1].

Let \(\varphi\) be a function in \(L^\infty(T)\) such that \(\lim_{n \to \infty} \tilde{T}_{\gamma_n} \varphi(x)\) fails to exist on a set \(E\), with \(|E| > 0\). Such a function exists by Theorem 1. Write \(\varphi(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}\). We can assume without loss of generality that \(\int_\pi \varphi(x) dx = 0\), and hence that \(a_0 = 0\). Define a function \(f\) on \(\mathbb{R}\) by

\[
\hat{f}(\xi) = \sum_{k=-\infty}^{\infty} a_k \chi_{(-k-1/2, -k+1/2)}(\xi).
\]

Thus

\[
(H_f f)^\sim(\xi) = \sum_{k=-\infty}^{\infty} a_k \chi_{(-k-1/2, -k+1/2)}(\xi) c_\gamma \text{sgn}(\xi)|\xi|^{i\gamma}.
\]

Taking inverse Fourier transforms, we have

\[
H_f f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (H_f f)^\sim(\xi) e^{-i\xi x} d\xi
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k \chi_{(-k-1/2, -k+1/2)}(\xi) c_\gamma \text{sgn}(\xi)|\xi|^{i\gamma} e^{-i\xi x} d\xi
\]

\[
= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} a_k c_\gamma \left[ \int_{-k-1/2}^{-k+1/2} \text{sgn}(\xi)(|\xi|^{i\gamma} - |k|^{i\gamma}) e^{-i\xi x} d\xi \right]
\]

\[
= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} a_k c_\gamma \text{sgn}(k)(|k|^{i\gamma} e^{ikx} \left( \frac{\sin(x/2)}{x/2} \right) + \hat{\epsilon}(x,\gamma),
\]

where \(\hat{\epsilon}(x, \gamma) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} a_k c_\gamma \int_{-k-1/2}^{-k+1/2} sgn(\xi)(|\xi|^{i\gamma} - |k|^{i\gamma}) e^{-i\xi x} d\xi\). Thus

\[
H_f f(x) = \left( \frac{\sin(x/2)}{x/2} \right) \tilde{T}_\gamma \varphi(x) + \hat{\epsilon}(x, \gamma).
\]
However,
\[
|e(x, \gamma)| \leq \frac{|c\gamma|}{2\pi} \sum_{k=-\infty}^{\infty} |a_k| \int_{-k-1/2}^{-k+1/2} |||\xi||^{i\gamma} - |k|^{i\gamma}|d\xi.
\]

By the mean value theorem, we can estimate the integral in this expression by
\[
\sup_{\xi \in (k-1/2, k+1/2)} |(e^{i\gamma \ln \xi})'| \frac{1}{2} \leq \frac{\gamma}{|k| - 1/2} \left(\frac{1}{2}\right).
\]
Consequently,
\[
|e(x, \gamma)| \leq \frac{|c\gamma|}{4\pi} \sum_{k=-\infty}^{\infty} |a_k| |k|^{\gamma/2} \leq \gamma \frac{|c\gamma|}{4\pi} \left(\sum_{k=-\infty}^{\infty} |a_k|^2\right)^{1/2} \left(\sum_{k=-\infty}^{\infty} \frac{1}{(|k| - 1/2)^2}\right)^{1/2} = O(\gamma).
\]

We now know
\[
H_{\gamma} f(x) = \left(\frac{\sin(x/2)}{x/2}\right) \tilde{T}_{\gamma} \varphi(x) + e(x, \gamma), \quad \text{with } e(x, \gamma) = O(\gamma).
\]

For each \( x \in E \), we know \( \lim_{n \to \infty} \tilde{T}_{\gamma_n} \varphi(x) \) will fail to exist. Since \( \lim_{n \to \infty} e(x, \gamma) = 0 \), we know that \( \lim_{n \to \infty} H_{\gamma_n} f(x) \) must also fail to exist on \( E \).

It remains to show that \( f \in \mathcal{L}^p(\mathbb{R}), 1 < p \leq \infty \). To see this we compute \( f \) explicitly, and find that
\[
f(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} a_k e^{ikx} \left(\frac{\sin x/2}{x/2}\right).
\]

Consequently
\[
\|f\|_p^p = \left(\frac{1}{2\pi}\right)^p \sum_{n=-\infty}^{n_0} \int_{2\pi n}^{2\pi(n+1)} \left|\frac{\sin x/2}{x/2}\right|^p \sum_{k=-\infty}^{\infty} |a_k e^{ikx}|^p dx.
\]

First note that \( |\sin x/2| \) is bounded by \( 2/n \) on \( [2\pi n, 2\pi(n+1)] \) if \( n \neq 0 \), or \( -1 \), and by \( 1 \) on \( [-2\pi, 0] \) and \( [0, 2\pi] \). Also note that
\[
\int_{2\pi n}^{2\pi(n+1)} \left|\sum_{k=-\infty}^{\infty} a_k e^{ikx}\right|^p dx
\]
is a constant \( b_p \) independent of \( n \). Consequently
\[
\|f\|_p^p \leq \left(\frac{1}{2\pi}\right)^p \left(2 + \sum_{n=-\infty}^{n_0} \frac{2^p}{|n|^p}\right) b_p < \infty. \quad \square
\]

Remark. Theorem 2 can be strengthened to give a function that is in all \( \mathcal{L}^p(\mathbb{R}), 0 < p \leq \infty \), such that almost everywhere convergence of \( \{H_{\gamma_n} f\} \) fails. The idea is to replace translates of the characteristic function of \((-1/2, 1/2)\) by a suitably smooth "bump function." See [1] for details.
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