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FRÉCHET VS. GÂTEAUX DIFFERENTIABILITY OF LIPSCHITZIAN FUNCTIONS

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ABSTRACT. Examples have been given of Lipschitzian functions that are Gâteaux-differentiable everywhere, but nowhere Fréchet-differentiable. One such example has been reported, mistakenly, in several papers as having domain in $L^2([0, \pi])$, when it should have been $L^1([0, \pi])$. We discuss this example.

The purpose of this note is to point out a misunderstanding that has been perpetuated about an example due to Sova [7]. In order to do this, we consider two mappings: (1) $f: L^1([0, \pi]) \to \mathbf{R}$ defined by $f(x) = \int_0^\pi \sin x(t)\, dt$, and (2) $g: L^2([0, \pi]) \to \mathbf{R}$ defined by $g(x) = \int_0^\pi \sin x(t)\, dt$. Clearly, g is the restriction of f to $L^2([0, \pi]) \subseteq L^1([0, \pi])$. The mapping f is an example of a Lipschitzian real-valued function that is everywhere Gâteaux-differentiable, but nowhere Fréchet-differentiable. In fact, f is a special case of a whole class of mappings, from the space $L^1(X, \Sigma, \mu)$ of all Σ-measurable, μ-integrable functions from X to $\mathbf{R}$, defined by Sova in [7] that are Gâteaux-differentiable, but not Fréchet-differentiable.

On the other hand, the function g is Fréchet-differentiable everywhere, but is given in some papers, in error, as an example of a mapping that is nowhere Fréchet-differentiable; cf. [3, p. 205], [4, p. 124], and [5, p. 125]. Earlier, Phelps [6, pp. 981–982] gave an example of an equivalent norm on l^1 that is Gâteaux-differentiable everywhere (except at the origin) and nowhere Fréchet-differentiable. Other examples of Lipschitzian real-valued functions that are nowhere Fréchet-differentiable were given by Aronszajn in [1]; his functions are on the space l^1.

In order to consider the differentiability of f and g, let $x, v \in L^1([0, \pi])$,

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$v \neq 0$, $h > 0$. Then

$$\begin{aligned}\lim_{h\downarrow 0}\frac{1}{h}\int_0^{\pi}&[\sin(x(t)+hv(t))-\sin x(t)]\,dt\\ &=\lim_{h\downarrow 0}\int_0^{\pi}\frac{\sin\frac{hv(t)}{2}}{\frac{h}{2}}\cos\left(x(t)+\frac{hv(t)}{2}\right)dt\\ &=\int_0^{\pi}v(t)\cos x(t)\,dt,\end{aligned}$$

since the integrand $((\sin\frac{hv(t)}{2})/(h/2))\cos(x(t)+\frac{hv(t)}{2})$ is dominated by $v \in L^1([0,\pi])$. Hence, the Gâteau x derivative of the mapping f at x is $D_G f(x) = \cos x$. It is clear that the mapping g, which is the restriction of f to $L^2([0,\pi])$, is also Gâteaux-differentiable and that the Gâteaux derivative $D_G g(x) = \cos x$ is a continuous mapping from $L^2([0,\pi])$ into $L^2([0,\pi])$ in the norm topologies. Therefore, g is Fréchet-differentiable everywhere (see [2, Examples 1 and 2, pp. 18–22] for a more general class having the two properties described previously). Actually, in our case, g is uniformly Fréchet-differentiable. (Apply Taylor's formula to the sine function, and conclude that $|\sin(a+b)-\sin a - b\cos a| = |\frac{b^2}{2}\sin z|$ for some z between a and $a+b$. Thus,

$$\int_0^{\pi}|\sin(x(t)+y(t))-\sin x(t)-y(t)\cos x(t)|\,dt \leq \int_0^{\pi}\frac{1}{2}y(t)^2\,dt = \frac{1}{2}||y||_2^2.$$

Hence, if $||y||_2 < 2\varepsilon$, we see that $|g(x+y)-g(x)-\langle y, \cos x\rangle| \leq \varepsilon||y||_2$.)

To prove that f is not Fréchet-differentiable at any point $x \in L^1([0,\pi])$, we will follow Sova's proof of [7, Theorem 2.1.6]. First, we show that for each $x \in L^1([0,\pi])$, there exists $v \in L^1([0,\pi])$ such that the Lebesgue measure of the set $\{t \in \mathbf{R} \mid 0 \leq t \leq \pi \text{ and } \sin(x(t)+v(t))-\sin x(t)-v(t)\cos x(t) \neq 0\}$ is positive. If not, let q be a rational number and define v_q by $v_q(t) = q$ for all $t \in [0,\pi]$. Then $v_q \in L^1([0,\pi])$ and the set $N_q = \{t \in [0,\pi] \mid \sin(x(t)+q)-\sin x(t) \neq q\cos x(t)\}$ has Lebesgue measure 0. Hence, the union $N = \bigcup\{N_q | q \text{ rational}\}$ also has measure 0. Thus, for all rational numbers q and all $t \notin N$, $\sin(x(t)+q)-\sin x(t) = q\cos x(t)$. This is a contradiction since the mapping $q\cos x(t)$ is a linear function of q, but $\sin(x(t)+q)-\sin x(t)$ is not.

Next, choose $v_0 \in L^1([0,\pi])$ such that $\mu(\{t \in [0,\pi] \mid \sin(x(t)+v_0(t)) - \sin x(t) - v_0(t)\cos x(t) \neq 0\}) > 0$, where μ denotes Lebesgue measure. Then we can find $\alpha > 0$ such that the set $Z = \{t \in [0,\pi] \mid \sin(x(t)+v_0(t)) - \sin x(t) - v_0(t)\cos x(t) > \alpha\}$ satisfies $\mu(Z) > 0$. Further, there exists a β and a measurable subset Z_0 of Z such that $\mu(Z_0) > 0$ and $|v_0(t)| < \beta$ for $t \in Z_0$. Choose a sequence $\{Z_n\}_{n=1}^{\infty}$ of measurable subsets of Z_0 of Z such that $Z_{n+1} \subset Z_n$, $\mu(Z_n) > 0$ for $n = 1, 2, \ldots$, and $\bigcap_{n=1}^{\infty} Z_n = \varnothing$, and define a sequence $\{h_n\}_{n=1}^{\infty}$ of functions in $L^1([0,\pi])$ by

$$h_n(t) = \begin{cases} v_0(t) & \text{if } t \in Z_n \\ 0 & \text{if } t \notin Z_n. \end{cases}$$

It is easy to check that $||h_n||_1 \to 0$ as $n \to \infty$, but

$$\frac{|\int_0^{\pi}[\sin(x(t)+h_n(t))-\sin x(t)-h_n(t)\cos x(t)]\,dt|}{||h_n||_1} \geq \frac{\alpha\mu(Z_n)}{\beta\mu(Z_n)} = \frac{\alpha}{\beta} > 0.$$

This shows that f is not Fréchet-differentiable at $x \in L^1([0, \pi])$.

Added in proof. It should be noted that the authors were not the first to discover the difficulties discussed in this paper. See R. R. Phelps, *Convex functions, monotone operators and differentiability*, Lecture Notes in Math., vol. 1364, Springer-Verlag, New York, 1989, p. 105. See also D. Preiss, *Frêchet derivatives of Lipschitz functions*, J. Funct. Anal. **91** (1990), 312–345, for a very strong positive result on differentiability of Lipschitz functions.

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