PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 114, Number 4, APRIL 1992

REAL ISOMETRIES BETWEEN JB^*-TRIPLES

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(Communicated by Paul S. Muhly)

ABSTRACT. It is shown that except for a certain class of JB^*-triples (for which the result is false), real linear surjective isometries preserve the triple product. In particular, unital real linear isometries of C^*-algebras are real linear Jordan *-isomorphisms.

1. INTRODUCTION AND PRELIMINARIES

A well-known result of Banach and Stone states that two compact Hausdorff spaces X, Y are homeomorphic iff $C_{\mathbf{R}}(X)$ and $C_{\mathbf{R}}(Y)$ are linearly isometric (where $C_{\mathbf{R}}(X)$ denotes the space of real-valued continuous functions on X). Equivalently, $C(X)$ and $C(Y)$ are *-isomorphic iff they are linearly isometric, where $C(X)$ denotes the C^*-algebra of all complex continuous functions on X. A noncommutative version of the Banach-Stone theorem was proved by Kadison [13]: Every surjective (complex) linear isometry between two unital C^*-algebras is a Jordan *-isomorphism followed by a left multiplication by a fixed unitary. The analogous result in the case of real C^*-algebras is an open question. However, as a consequence of the main result of this paper, Kadison's theorem remains true for real-linear isometries.

Kadison's result was later generalized to various objects, namely J^*-algebras (Harris [12]), JB^*-algebras (Wright-Youngson [19]), and most recently JB^*-triples, which are common generalization of all of the above (Kaup [15]). JB^*-triples are certain complex Banach spaces equipped with a triple product instead of a binary one. They appear as the ranges of contractive projections on C^*- and JB^*-algebras and in the study of bounded symmetric domains in Banach spaces. For concrete examples of JB^*-triples we can consider J^*-algebras (introduced by Harris). These are norm-closed subspaces of $B(H, K)$, the bounded operators between two Hilbert spaces, closed under the triple product $\{x, y, z\} = \frac{1}{2}(xy^*z+zy^*x)$. The precise definition as well as general properties of JB^*-triples can be found in [17] and [18]. Further background information can be found in [4, 9, 10, 14, 16]. The above-mentioned theorem of Kaup states that the complex triple isomorphisms between two JB^*-triples are exactly the surjective linear isometries.

Received by the editors June 11, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46H15.
Presented at the GPOTS, Houston, May 1989.

In [2], it is shown that to each nonzero functional on a JB^*-triple there corresponds a unique representation into a JBW^*-triple (i.e., a JB^*-triple which has a predual). Consequently, a natural candidate for the "state space" of a JB^*-triple is the entire unit sphere of its dual. Motivated by the symmetry transformation in quantum mechanics, one would like to consider invertible affine maps on the unit sphere of the dual of a JB^*-triple. Unlike the situation in C^*-algebras and JB^*-algebras, these maps turn out to be the adjoints of real (not complex) linear surjective isometries. In this paper, we study such maps. We show that except for a certain set of JB^*-triples (which does not include any C^*-algebras, and for which the result is false), each real linear surjective isometry is the sum of a complex linear and a complex conjugate linear isometry. Such isometries preserve the triple product. Thus, this result can be viewed as an extension of a theorem of Wigner (see [3, §3.2.1]) or of the above-mentioned theorem of Kaup. It also provides a partial converse to a theorem of Friedman and Hakeda [8].

Following the notation in [9], we will denote the triple product as $\{x, y, z\}$, the cube $\{x, x, x\}$ as x^3. Two elements x, y are *orthogonal* ($x \perp y$) if $\{x, y, z\} = 0$ for all z. An element e is called a *tripotent* if $e^3 = e \neq 0$. The Peirce projections associated to a tripotent e are denoted as $P_k(e)$, ($k = 0, 1, 2$), their ranges on a JB^*-triple U as $U_k(e)$. These are subspaces of U characterized by the following property: $x \in U_k(e)$ iff $\{e, e, x\} = \frac{1}{2}k\,x$. The Peirce space $U_2(e)$ is a JB^*-algebra, and we have:

$$U = U_2(e) \oplus U_1(e) \oplus U_0(e);$$
$$\{U_i(e), U_j(e), U_k(e)\} \subseteq U_{i-j+k}(e)$$

where $U_l(e) := \{0\}$ if $l \notin \{0, 1, 2\}$; and $\{U_2(e), U_0(e), U\} = \{U_0(e), U_2(e), U\} = \{0\}$. In case U is a JBW^*-triple, elements x, y are orthogonal exactly when there is a tripotent e such that $x \in U_2(e)$ and $y \in U_0(e)$. Our first step in this paper is to show that the cube of an element and the orthogonality of two elements are preserved under real linear surjective isometries.

Proposition 1.1. *Let M, N be JB^*-triples and $\phi: M \to N$ be a real linear surjective isometry. Then, for all x, y in M,*

$$\phi(x^3) = \phi(x)^3$$

and

$$x \perp y \quad \textit{iff}\ \phi(x) \perp \phi(y).$$

The key ingredient in the proof is the one-to-one correspondence between the tripotents of a JBW^*-triple U and the norm-exposed faces of the unit ball U_{*1} of its predual, where orthogonal tripotents corresponds to orthogonal faces. This idea was initiated by Friedman and Russo in [11] and explicitly discussed in [5].

Recall that by a *norm-exposed face* of U_{*1}, one means a set of the form

$$F_x = \{\psi \in U_{*1} : \psi(x) = 1\}, \qquad x \in U, \quad +\|x\| = 1.$$

F_x is indeed a face of U_{*1} in the sense that if a nontrivial convex combination of two functionals f, g in U_{*1} is contained in F_x, then both f and g are contained in F_x.

Two functionals f and g in U_* are said to be *orthogonal* ($f \diamond g$) if they satisfy one of the following equivalent conditions (cf. [11, Proposition 1.1]):

(i) $\|f \pm g\| = \|f\| + \|g\|$.

(ii) There exist u, v in U such that $\|u\| = \|v\| = 1$ and $f(u) = \|f\|$, $f(v) = 0$, $g(v) = \|g\|$, and $g(u) = 0$.

Two norm-exposed faces F_x and F_y are *orthogonal* if $f \diamond g$ for all $(f, g) \in F_x \times F_y$. The set of all functionals orthogonal to F_x will be denoted as $F_x^\diamond$.

Except for the injectivity of the map $u \mapsto F_u$, the following two facts are proved in [5]:

(i) [5, Lemmas III, IV]. The map $u \mapsto F_u$, sets up a one-to-one correspondence between the tripotents of U and the norm-exposed faces of U_{*1}. Moreover, two tripotents u and v are orthogonal iff their corresponding faces F_u, F_v are orthogonal.

(ii) [5, Lemma V]. An element u in U is a tripotent iff $\|u\| = 1$ and $\langle u, F_u^\diamond \rangle = 0$.

To see that the map $u \mapsto F_u$ in (i) is one-to-one, one notes that by the Jordan decomposition of Hermitian functionals on a JB^*-algebra and [5, Lemma Ib], $P_2(u)^*(U_*) = sp_{\mathbf{C}} F_u$. Since $u \in U_2(u) = P_2(u)U$, u is determined by its values on $\mathrm{sp}_{\mathbf{C}} F_u$. Thus if $F_u = F_w$, then $u = w$. The proof of (i) also appears in [7].

The following notation will be used in the coming proof: If X is a complex Banach space, we will let $X_{\mathbf{R}}^*$ denote its real dual (i.e., the space of real-valued bounded real-linear functionals on X). If $f \in X^*$, we will denote its real part as $\mathrm{Re}(f)$, so that $\langle \mathrm{Re}(f), x \rangle = \mathrm{Re}\langle f, x \rangle$. If $g \in X_{\mathbf{R}}^*$, we will denote its complexification by $\Phi(g)$, so that $\langle \Phi(g), x \rangle = \langle g, x \rangle - i\langle g, ix \rangle$. It is well known that the map $f \mapsto \mathrm{Re}(f)$ is a real linear isometry between X^* and $X\mathbf{R}^*$, with inverse Φ.

Proof of Proposition 1.1. By [1] and [6], the biduals M^{**} and N^{**} are JBW^*-triples that contain M and N as subtriples. Let $\phi^*: N_{\mathbf{R}}^* \to M_{\mathbf{R}}^*$ be the (real) adjoint of ϕ. We define a map $\psi: N^* \to M^*$ by $\psi(f) = \Phi[\phi^* \mathrm{Re}(f)]$. Since ψ is a real linear isometry, we can repeat the process to define a map $\tilde{\phi}: M^{**} \to N^{**}$ by $\tilde{\phi}(f) = \Phi[\psi^* \mathrm{Re}(f)]$. The map $\tilde{\phi}$ is then a real linear isometry extending ϕ.

Let $x \in M^{**}$, $\|x\| = 1$; we can verify that

$$F_{\tilde{\phi}(x)} = \psi^{-1}(F_x). \tag{1.1}$$

Thus, ψ maps norm-exposed faces to norm-exposed faces. If p is a tripotent in M^{**}, then $\langle p, F_p^\diamond \rangle = 0$. Using (1.1), we can verify that $\langle \tilde{\phi}(p), F_{\tilde{\phi}(p)}^\diamond \rangle = 0$. This shows that $\tilde{\phi}(p)$ is a tripotent. If q is another tripotent orhtogonal to p, then their associated faces F_p and F_q are orthogonal. Since, as a real linear isometry, ψ^{-1} preserves the orthogonality of functionals, $F_{\tilde{\phi}(p)}$ and $F_{\tilde{\phi}(q)}$ are orthogonal, implying $\tilde{\phi}(p) \perp \tilde{\phi}(q)$. Thus, orthogonal tripotents are mapped to orthogonal tripotents under $\tilde{\phi}$. For any $\varepsilon > 0$, there are orthogonal tripotents $u_1, u_2, \ldots, u_n$ and positive scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\|x - y\| < \varepsilon$ and $\|y\| \leq \|x\|$, where $y = \sum_{i=1}^n \lambda_i u_i$. It follows that $\|\tilde{\phi}(x) - \sum \lambda_i \tilde{\phi}(u_i)\| < \varepsilon$ and $\|x^3 - \sum \lambda_i^3 u_i\| < 3\varepsilon\|x\|^2$, $\|\tilde{\phi}(x)^3 - \sum \lambda_i^3 \tilde{\phi}(u_i)\| < 3\varepsilon\|x\|^2$. Therefore, $\|\tilde{\phi}(x^3) - \tilde{\phi}(x)^3\| < 6\varepsilon\|x\|^2$, and since ε is arbitrary, the first conclusion follows.

The second statement follows by approximating x and y in the orthogonal JBW^*-triples $U_2(e)$ and $U_0(e)$ containing x and y, respectively. □

If the isometry ϕ were complex linear, we could apply the polarization identity

$$\{x, y, z\} = \frac{1}{8} \sum_{\substack{\alpha^4=1 \\ \beta^2=1}} \alpha\beta(x + \alpha y + \beta z)^3$$

to obtain Kaup's theorem. The following corollary was proved directly by Kaup with the technique of holomorphy.

Corollary 1.2. *If ϕ is a complex linear or complex conjugate linear surjective isometry, then ϕ preserves the triple product.*

Since our ϕ is known only to be real linear, the polarizaton is no longer applicable. Thus, a new technique is needed. Since each JB^*-triple can be embedded into a l^∞-sum of Cartan factors (The Gelfand-Naimark Theorem, [10]), our next step is to analyze real isometries between Cartan factors. This is the topic of the next section. A brief description of Cartan factors can be found in [4]; for full detail, see [14].

2. Real isometries between Cartan factors

Our goal in this section is to show that a real linear surjective isometry between Cartan factors is either a (complex) linear or a conjugate linear homomorphism (Proposition 2.6). Each Cartan factor is spanned by a grid which consists of quadrangles or trangles being "glued" together in a certain way ([4], [6]). Therefore we will first show that the isometry, when restricted to the span of each quadrangle or each trangle, is either linear or conjugate linear. Then, it follows that the map is either linear or conjugate linear on the whole factor.

A tripotent e in a *JB^*-triple U* is *minimal* if $U_2(e) = \mathbf{C}e$ or, equivalently, if e is not the sum of orthogonal tripotents. An element has *rank n* if it is a linear combination of n pairwise orthogonal minimal tripotents. If U has a maximal family consisting of n pairwise orthogonal minimal tripotents, we say $\operatorname{rank} U = n$. Note that as a consequence of Proposition 1.1, the rank of an element is preserved by a surjective linear isometry ϕ. In particular, $\phi(e)$ is a minimal tripotent if e is a minimal tripotent. The space $B(H, K)$ is of rank 1 if H or K is one dimensional.

We say that two tripotents u and v of U are *colinear* $(u \top v)$ if $u \in U_1(v)$ and $v \in U_1(u)$. We say v *governs* $u(v \vdash u)$ if $u \in U_2(v)$ and $v \in U_1(u)$.

A quadruplet of tripotents (u_1, u_2, u_3, u_4) is called a *quadrangle* if $u_i \top u_{i+1}$, $u_i \perp u_{i+2}$, and $2\{u_i, u_{i+1}, u_{i+2}\} = u_{i+3}$. (The indices are computed modulo 4.)

A triplet (u_1, u_2, u_3) is called a *prequadrangle* if $u_1 \top u_2 \top u_3$ and $u_1 \perp u_3$. Naturally such a prequadrangle can be completed into a quadrangle (u_1, u_2, u_3, u_4) with $u_4 = 2\{u_1, u_2, u_3\}$.

A triplet $(u, v, \tilde{u})$ is called a *trangle* if $u \perp \tilde{u}$, $v \vdash u$, and $\{v, u, v\} = \tilde{u}$.

A pair (u, v) is called a *pretrangle* if $v \vdash u$. Such a pretrangle can be completed into a trangle $(u, v, \tilde{u})$ with $\tilde{u} = \{v, u, v\}$.

As concrete examples of quadrangles and trangles, we can consider (e_{ij}, e_{il}, e_{kj}) and $(e_{ii}.e_{ij} + e_{ji}, e_{jj})$, where i, j, k, l are distinct indices, the e_{ij}'s are matrix units, and the triple product is $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$.

Lemma 2.1. *Let u be a minimal tripotent in a Cartan factor U. If $U_0(u) \neq \{0\}$, then $U_2(u) = U_0(u)^\perp$.*

Proof. It is obvious that $U_2(u) \subseteq U_0(u)^\perp$. To show the equality, it suffices to show that $P_1(u)x = 0$ whenever $x \in U_0(u)^\perp$. Let $z = P_1(u)x$. Since $x = P_2(u)x + z$ is orthogonal to $U_0(u)$ and $\{U, U_0(u), U_2(u)\} = 0$, we have

$$\{f, f, z\} = 0 \quad \forall f \in U_0(u). \tag{2.1}$$

From the classification scheme in [4, Proposition 2.1 and its Corollary], the rank of $U_1(u)$ is either 1 or 2. If $\operatorname{rank} U_1(u) = 1$, there is a tripotent W such that $w \vdash u$ and $z = \alpha w$. Let $\tilde{u} = \{w, u, w\}$. Then $(u, w, \tilde{u})$ form a trangle [4, Proposition 2.1(ii)]. This implies that $\{\tilde{u}, \tilde{u}, z\} = \frac{1}{2}z$, contradicting (2.1). If $\operatorname{rank} U_1(u) = 2$, we can find tripotents w, $\tilde{w}$ in $U_1(u)$ such that $(w, u, \tilde{w})$ form a prequadrangle [4, Proposition 2.1], and $z = \alpha_1 w + \alpha_2 \tilde{w}$. Let $\tilde{u} = 2\{w, u, \tilde{w}\}$. Then, by [4, Proposition 1.7], $(u, w, \tilde{u}, \tilde{w})$ form a quadrangle, implying $\{\tilde{u}, \tilde{u}, z\} = \frac{1}{2}z$, again contradicting (2.1). This shows that $z = 0$. □

In Lemmas 2.2–2.5, U, V denote Cartan factors, and ϕ denotes real linear isometry from U onto V.

Lemma 2.2. *If u is a minimal tripotent with $U_0(u) \neq \{0\}$, then $\phi(iu) = \pm i\phi(u)$.*

Proof. Since u is minimal, $\phi(u)$ is minimal; that is, $V_2(\phi(u)) = \mathbf{C}\phi(u)$. On the other hand, from Lemma 2.1 it follows that $\phi(U_2(u)) = V_2(\phi(u))$; in particular, $\phi(iu) = \lambda\phi(u)$ for some $\lambda \in \mathbf{C}$ with $|\lambda| = 1$. From $\|\phi(iu) - \phi(u)\| = \|iu - u\| = \sqrt{2}$, it follows that $\phi(iu) = \pm i\phi(u)$. □

In Lemma 2.4, we will perform calculations on the Cartan factor known as the spin factor. A JBW^*-triple U is a spin factor if $\dim U \geq 3$ and there are two orthogonal minimal tripotents u, $\tilde{u}$ in U such that $U = U_2(u + \tilde{u})$. Spin factors are generalizations of $M_2(\mathbf{C})$ (the space of 2×2 complex matrices) and $S_2(\mathbf{C})$ (the space of 2×2 symmetric complex matrices). The structure of a spin factor is more conveniently described in terms of odd quadrangles and odd trangles, which are cosmetic modifications of the quadrangles and trangles described earlier: $(u, v, \tilde{u}, \tilde{v})$ is an *odd quadrangle* if $(u, v, \tilde{u}, -\tilde{v})$ is a quadrangle; $(u, v, \tilde{u})$ is an *odd trangle* if $(u, v, -\tilde{u})$ is a trangle.

It is known that each spin factor is the norm-closed span of a family of tripotents $\{u_i, \tilde{u}_i, u_0 \colon i \in I\}$ called a spin-grid. That is, for $i \neq j$, $(u_i, u_j, \tilde{u}_i, \tilde{u}_j)$ form an odd quadrangle and $(u_i, u_0, \tilde{u}_i)$ form and odd trangle. As with matrices, we can define the determinant and the Hilbert-Schmidt norm $\|\cdot\|_2$ of an element (relative to a given spin-grid) as follows: For $x = \sum_{i \in I} a_i u_i + \tilde{a}_i \tilde{u}_i + a_0 u_0$, let

$$\det x := \sum a_i \tilde{a}_i + a_0^2,$$

$$\|x\|_2 := \left(\sum |a_i|^2 + |\tilde{a}_i|^2 + 2|a_0|^2\right)^{1/2}.$$

The following are equivalent: (i) $\operatorname{rank} x = 1$; (ii) $\det x = 0$; (iii) $\|x\| = \|x\|_2$. A more detailed discussion of the above can be found in [4, §3].

The following lemma is used in the proof of Lemma 2.4. Recall that there is a natural order among the tripotents: $e < f$ if $f - e$ is a tripotent orthogonal to e.

Lemma 2.3. *If u and $\tilde{u}$ are orthogonal minimal tripotents of U, then* $\phi(U_2(u+\tilde{u})) = V_2(\phi(u)+\phi(\tilde{u}))$.

Proof. Since $U_2(u+\tilde{u})$ is norm-spanned by quadrangles and trangles, it suffices to show that ϕ maps these quadrangles and trangles into $V_2(\phi(u)+\phi(\tilde{u}))$. By considering ϕ^{-1}, the lemma will follow.

If $(u, v, \tilde{u}, \tilde{v})$ form a quadrangle, and $e = \frac{1}{2}(u+\tilde{u}+v+\tilde{v})$, $f = \frac{1}{2}(u+\tilde{u}-v-\tilde{v})$, then e and f are orthogonal tripotents with $e+f = u+\tilde{u}$. Thus $e < u+\tilde{u}$ and $f < u+\tilde{u}$. Since ϕ preserves the order among the tripotents, we have $\phi(e) < \phi(u)+\phi(\tilde{u})$, $\phi(f) < \phi(u)+\phi(\tilde{u})$. This implies that $\phi(v)+\phi(\tilde{v}) = \phi(e)-\phi(f)$ is contained in $V_2(\phi(u)+\phi(\tilde{u}))$. Since $\phi(v)$ and $\phi(\tilde{v})$ are orthogonal, each of them is contained in $V_2(\phi(u)+\phi(\tilde{u}))$.

If $(u, v, \tilde{u})$ form a trangle, let $e = \frac{1}{2}(u+\tilde{u}+v)$, $f = \frac{1}{2}(u+\tilde{u}-v)$; then e and f are orthogonal tripotents with $e+f = u+\tilde{u}$. An argument similar to the one above shows that $\phi(v)$ is contained in $V_2(\phi(u)+\phi(\tilde{u}))$. □

Lemma 2.4. *Let u, $\tilde{u}$, be orthogonal minimal tripotents of U.*

(i) *If $(u, v, \tilde{u}, \tilde{v})$ is a quadrangle in U, then $(\phi(u), \phi(v), \phi(\tilde{u}), \phi(\tilde{v}))$ is a quadrangle in V, and ϕ is either linear or conjugate linear on the (complex) span* of $\{u, v, \tilde{u}, \tilde{v}\}$.

(ii) If $(u, v, \tilde{u})$ is a trangle in U then $(\phi(u), \phi(v), \phi(\tilde{u}))$ is a trangle in V, and ϕ is either linear or conjugate linear on the span of $\{u, v, \tilde{u}\}$.

Proof. Because of Lemma 2.3, we can assume without loss of generality that U and V are spin factors.

(i) Assume $(u, v, \tilde{u}, \tilde{v})$ is a quadrangle. Since V is a spin factor whose dimension is at least 4, $\operatorname{rank} V_1(\phi(u)) \cap V_1(\phi(\tilde{u})) = 2$, and thus we can choose tripotents w, $\tilde{w}$ in V such that $(\phi(u), w, \phi(\tilde{u}), \tilde{w})$ form a quadrangle [4, Proposition 2.1] with $\phi(v) = a\phi(u)+bw+c\phi(\tilde{u})+d\tilde{w}$. Since v, $u+v$, $u \pm iv$ are of rank 1, their images $\phi(v)$, $\phi(u)+\phi(v)$, $\phi(u) \pm i\phi(v)$ are also of rank 1. From the fact that the norm of a rank 1 element coincides with its Hilbert-Schmidt norm follow

$$|a|^2+|b|^2+|c|^2+|d|^2 = \|\phi(v)\|^2 = \|v\|^2 = 1,$$
$$|a+1|^2+|b|^2+|c|^2+|d|^2 = \|\phi(u)+\phi(v)\|^2 = \|u+v\|^2 = 2,$$
$$|a+i|^2+|b|^2+|c|^2+|d|^2 = \|\phi(u+iv)\|^2 = \|u \pm iv\|^2 = 2,$$

which imply $a = 0$. By symmetry we also have $c = 0$. From $\det\phi(v) = 0$ follows $b = 0$ or $d = 0$. Thus, we can assume without loss of generality that $\phi(v) = w$. Since $V_0(w) = \mathbf{C}\tilde{w}$, $\phi(v) = \lambda\tilde{w}$ for some λ, with $|\lambda| = 1$. Let $z = u+v+\tilde{u}+\tilde{v}$; then $\phi(z) = \phi(u)+w+\phi(\tilde{u})+\lambda\tilde{w}$. Using the fact that $\phi(z^3) = \phi(z)^3$, we infer $\lambda = 1$.

It remains to show that ϕ is either linear or conjugate linear on $\operatorname{span}\{u, v, \tilde{u}, \tilde{v}\}$. Let $z = au+bv+c\tilde{v}+d\tilde{u}$, where a, b, c, d are arbitrary complex numbers. From Lemma 2.2, it follows that $\phi(z) = \rho_1(a)\phi(u)+\rho_2(b)\phi(v)+\rho_3(c)\phi(\tilde{v})+\rho_4(d)\phi(\tilde{u})$, where each ρ_i is either the identity map or the conjugation map on $\mathbf{C}$. Since $\det\phi(z) = 0$ iff $\det z = 0$, we have

$\rho_1(a)\rho_4(d) - \rho_2(b)\rho_3(c) = 0$ iff $ad - bc = 0$. This shows ϕ is either linear or conjugate linear on span $\{u, v, \tilde{u}, \tilde{v}\}$.

(ii) Let $(u, v, \tilde{u})$ be a trangle in U. If $\dim U \geq 4$, there are tripotents e, $\tilde{e}$ such that $(u, e, \tilde{u}, \tilde{e})$ form a quadrangle and $v = e + \tilde{e}$. The desired conclusion then follows from part (i). Thus we can assume $\dim U = \dim V = 3$. Let w be a tripotent in V such that $(\phi(u), w, \phi(\tilde{u}))$ form a trangle, and let $\phi(v) = a\phi(u) + bw + d\phi(\tilde{u})$ for some numbers a, b, d. Let α, r, γ be real numbers, and let $z = \alpha u + rv + \gamma\tilde{u}$. Since $\det z = 0$ iff $\det\phi(z) = 0$, we have $\alpha\gamma - r^2 = 0$ iff $(\alpha + ra)(\gamma + rd) - r^2 b = 0$, which implies that $a = d = 0$ and $b^2 = 1$. This shows that $(\phi(u), \phi(v), \phi(\tilde{u}))$ form a trangle.

Next, we show $\phi(iv) = \pm i\phi(v)$. Let $\phi(iv) = a\phi(u) + b\phi(v) + d\phi(\tilde{u})$ for some a, b, d, and $z = \alpha u + r(iv) + \gamma\tilde{u}$ for some real numbers α, r, γ. Since $\det z = 0$ iff $\det\phi(z) = 0$, we have $a = d = 0$ and $b = \pm i$. The same argument as in part (i) shows that ϕ is either linear or conjugate linear on span $\{u, v, \tilde{u}\}$. □

Lemma 2.5. *If* rank $U \geq 2$, *then* ϕ *is* w^**-continuous.*

Proof. Let $\{x_\alpha\}$ be a net in U. Since U is the w^*-closed span of its minimal tripotents, from [9, Proposition 4 and Theorems 1 and 2] it follows that $x_\alpha \xrightarrow{w^*} 0$ iff $P_2(u)x_\alpha \to 0$ for every minimal tripotent u of U. For the same reason, $\phi(x_\alpha) \xrightarrow{w^*} 0$ in V iff $P_2(w)\phi(x_\alpha) \to 0$ for every minimal tripotent w in V. This, together with Lemma 2.1, implies that ϕ is w^*-continuous. □

Since each Cartan factor, except those of rank 1, is the w^*-closed span of a grid built up from quadrangles and trangles, the next proposition follows immediately from Lemmas 2.4 and 2.5.

Proposition 2.6. *Let U, V be Cartan factors with* rank $U \geq 2$ *and $\phi: U \to V$ be a real linear surjective isometry. Then ϕ is either* (*complex*) *linear or conjugate linear on U. Moreover, ϕ preserves the triple product, i.e.,* $\phi\{x, y, z\} = \{\phi(x), \phi(y), \phi(z)\}$.

Remark 2.7. The conclusion of Proposition 2.6 is false if rank $U = 1$.

For example, let $U = M_{1,2}(\mathbf{C})$ with the triple product defined as $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$. Since rank $U = 1$, the norm on U is exactly the Hilbert space norm of $\mathbf{C}^2$ ([4, Case 1]). That is $\|(a, b)\|^2 = |a|^2 + |b|^2$. Let $\phi: U \to U$ be defined as $\phi(\alpha + i\beta, \gamma + i\delta) = (\alpha + i\gamma, \beta + i\delta)$; then ϕ is a real linear isometry. However, ϕ does not preserve the triple product on U. For instance, let $x = (1 + i, 0)$, $y = (0, 1)$; then $\phi\{x, y, x\} = 0$, while $\{\phi(x), \phi(y), \phi(x)\} = -(i, i)$.

3. Real isometries between JB^*-triples

When we say M is the direct sum of subtriples M_1 and M_2, it is understood that M_1 and M_2 are orthogonal as subtriples of M. We will call a triple system *trivial* if it is one-dimensional.

Theorem 3.1. *Let M, N be JB^*-triples and $\phi: M \to N$ be a real linear surjective isometry. If M^{**} does not have a nontrivial Cartan factor of rank* 1 *as a summand, then M is the direct sum of JB^*-subtriples M_1, M_2 such that $\phi|M_1$ is a* (*complex*) *linear and $\phi|M_2$ is a conjugate linear homomorphism.*

Proof. Let $M^{**} = M_a^{**} \oplus_{l_\infty} M_n^{**}$ and $N^{**} = N_a^{**} \oplus_{l_\infty} N_n^{**}$ be the decompositions of M^{**} and N^{**} into atomic and nonatomic parts ([9, Theorem 1]). Let $\tilde{\phi}: M^{**} \to N^{**}$ be the extension of ϕ defined as in the proof of Proposition 1.1. Since M_a^{**} is the w^*-closed real span of the minimal tripotents of M^{**}, we have

$$\tilde{\phi}(M_a^{**}) = N_a^{**}, \qquad \tilde{\phi}(M_n^{**}) = N_n^{**}.$$

By [10, Proposition 2], M_a^{**} and N_a^{**} are the direct sums of (pairwise orthogonal) Cartan factors. Then Propositions 1.1 and 2.6 imply that M_a^{**} is the direct sum of (orthogonal) subtriples U_1 and U_2 such that $\tilde{\phi}$ is linear on U_1, conjugate linear on U_2.

Let $\pi_1: M^{**} \to M_a^{**}$ and $\pi_2: N^{**} \to N_a^{**}$ be the projections onto the atomic parts. By ([10, Proposition 1]), $\pi_1|M$ and $\pi_2|N$ are isometric (complex) homomorphisms, implying that

$$\tilde{\phi}\pi_1(x) = \pi_2\phi(x), \quad \forall x \in M. \tag{3.1}$$

If $M_1 = \{x \in M: \pi_1(x) \in U_1\}$, $M_2 = \{x \in M: \pi_1(x) \in U_2\}$, then M_1 and M_2 are orthogonal subtriples of M, and the map ϕ is linear on M_1, conjugate linear on M_2.

Let us define the maps ϕ_1 and ϕ_2 on M as follows:

$$\phi_1(x) = \frac{i\phi(x) + \phi(ix)}{2i}, \qquad \phi_2(x) = \frac{i\phi(x) - \phi(ix)}{2i}.$$

Then ϕ_1 is linear, ϕ_2 is conjugate linear, and $\phi = \phi_1 + \phi_2$.

Define the maps $\tilde{\phi}_1$ and $\tilde{\phi}_2$ on M_a^{**} as follows:

$$\begin{aligned} \tilde{\phi}_1 &= \tilde{\phi} \quad \text{on } U_1, \qquad & \tilde{\phi}_1 &= 0 \quad \text{on } U_2, \\ \tilde{\phi}_2 &= \tilde{\phi} \quad \text{on } U_2, \qquad & \tilde{\phi}_2 &= 0 \quad \text{on } U_1. \end{aligned}$$

Then $\tilde{\phi}_1$ is linear, $\tilde{\phi}_2$ is conjugate linear, and $\tilde{\phi}|M_a^{**} = \tilde{\phi}_1 + \tilde{\phi}_2$. From (3.1) it follows that

$$\tilde{\phi}_1\pi_1(x) = \pi_2\phi_1(x), \qquad \tilde{\phi}_2\pi_1(x) = \pi\tilde{\phi}_2(x)$$

for all $x \in M$. A diagram-chasing type argument then shows that $M_1 + M_2 = M$ and $M_1 \cap M_2 = \{0\}$. $\square$

Remark. Since every w^*-dense representation of M can be realized in M^{**} ([2, Proposition 6]), M^{**} has no nontrivial Cartan factor of rank 1 as a summand iff M has no w^*-dense representation into a nontrivial Cartan factor of rank 1.

Next, we look at real isometries between JB^*-algebras or C^*-algebras. Recall that each JB^*-algebra is a JB^*-triple under the triple product

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*. \tag{3.2}$$

Corollary 3.2. *Let A, B be JB^*-algebras and $\phi: A \to B$ be a real linear surjective isometry. Then A is the direct sum of JB^*-subalgebras A_1, A_2 such that $\phi|A_1$ is a linear and $\phi|A_2$ is a conjugate linear triple homomorphism. Moreover, if A, B are unital and $\phi(1_A) = 1_B$, then the involution and the binary product on A are preserved under ϕ.*

Proof. The bidual A^{**} is a JB^*-algebra, and it has an identity 1_A. Since $\{1_A, 1_A, x\} = x$ for all $x \in A^{**}$, A^{**} cannot have a nontrivial summand of rank 1. Thus, by Theorem 3.1, A is a direct sum of JB^*-subtriples A_1, A_2 with $\phi|A_1$ a linear and $\phi|A_2$ a conjugate linear triple homomorphism. Using (3.2) and the existence of an approximate identity in the algebra A, we can verify that each A_i is a *-subalgebra of A. The rest of the proof follows from (3.2). □

If A is a C^*-algebra, it is a JB^*-triple under $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$, and any two elements x, y of A are orthogonal in the algebraic sense (i.e., $xy^* = y^*x = 0$) iff they are orthogonal in the triple sense (i.e., $\{x, y, z\} = 0$, $\forall z \in A$). Thus, A is the direct sum of A_1 and A_2 as a C^*-algebra iff it is the direct sum of A_1 and A_2 as a triple system. Moreover, an element u in the C^*-algebra A is unitary iff $\{u, u, z\} = z$ for all z in A. Consequently, we have the following version of Corollary 3.2 for C^*-algebras, which extends Kadison's theorem:

Corollary 3.3. *Let A, B be C^*-algebras and $\phi: A \to B$ be a real linear surjective isometry. Then A is the direct sum of C^*-subalgebras A_1, A_2 such that $\phi_1 := \phi|A_1$ is a linear and $\phi_2 := \phi|A_2$ is a conjugate linear triple homomorphism. Moreover, if A is unital then (for each i), $\phi_i = \mu_i\psi_i$, where ψ_i is a Jordan *-homomorphism and μ_i is the multiplication by a unitary element in the C^*-algebra $\phi_i(A_i)$.*

Acknowledgment

The contents of this paper is a part of the author's Ph.D. work under the supervision of Professor B. Russo at the University of California, Irvine. The proof of Lemma 2.3 was suggested by Professor Y. Friedman. The author is grateful to both Professors Russo and Friedman for their guidance.

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