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GL(2, Z) ACTION ON A TWO TORUS

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ABSTRACT. We study the group $\mathrm{GL}(2, Z)$ action on a two torus T^2 with Lebesgue measure. We show that any measure-preserving transformation that commutes with the group action is the action of a matrix of the form $\left(\begin{smallmatrix} m & 0 \\ 0 & m \end{smallmatrix}\right)$, where m is an integer. We also show that all factors come from these commuting transformations. Finally we show that the set of self-joinings consists of the product measure and the measures sitting on the graphs $\{(Ku, Mu) : K = \left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right),\ M = \left(\begin{smallmatrix} m & 0 \\ 0 & m \end{smallmatrix}\right)$, and $u \in T^2\}$. We provide an example whose self-joinings consist only of the product measure and the diagonal measure.

0. INTRODUCTION

We let $H = \mathrm{GL}(2, Z)$. We consider the group H action on a two torus $(T^2, \mathscr{F}, \nu)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$

where $\mathscr{F}$ denotes the product σ-algebra and ν denotes the product measure $\mu \times \mu$ on $T^2 = \{(x, y); x, y \in S^1\} = S^1 \times S^1$. We denote this dynamical system by $(T^2, H, \mathscr{F}, \nu)$. It is well known that H has many subgroups, each of which is isomorphic to Z and gives rise to a Bernoulli action. Namely, if a matrix $M \in H$ has an eigenvalue whose absolute value is greater than 1, then the action of M on T^2 is a Bernoulli action [K].

This work is motivated by the conjecture of del Junco and Rudolph in [JRu] that there exists no measure-preserving transformation except the identity map which commutes with the group action. More precisely, they conjectured that the set of self-joinings of the group action consists only of the product measure $\nu \times \nu$ and the diagonal measure. By a self-joining here we mean a twofold self-joining, which is an invariant and ergodic measure λ on $T^2 \times T^2$ under the diagonal group action $H \times H$ with the right marginals. That is, it satisfies $\lambda(T^2 \times A) = \lambda(A \times T^2) = \nu(A)$ for any measurable subset $A \subset T^2$.

The group H action on T^2 is different in many ways from the examples whose self-joinings have been investigated. First of all, the group H acts not

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only measurably but "algebraically" on $(T^2, \mathscr{F}, \nu)$. Second, H is not a commutative group. Since H is not an amenable group, the self-joinings cannot be investigated using the pointwise ergodic theorem or the mixing property like most other examples. (Most examples whose self-joinings are investigated so far are Z- or R-actions. A class of Z^2-actions with the property of minimal self-joinings (MSJ) are studied in [PRo].)

We sometimes identify a matrix with the action of the matrix on T^2. We define the centralizer of a group G action on a measure space, denoted by $\mathscr{L}(G)$, to be the set of all measure-preserving transformations that commute with every element of G. Clearly the actions of the matrices of the form $\left(\begin{smallmatrix} m & 0 \\ 0 & m \end{smallmatrix}\right)$, where M is an integer, commute with every element of the group H, and they are measure preserving. (They are not one-to-one unless $m = \pm 1$.) Hence they are contained in $\mathscr{L}(H)$, which contradicts the conjecture in [JRu]. We study the group action via its centralizer (§1), its factors (§2), and its self-joinings (§3). (Although the self-joinings help us understand the centralizer and the factors, we are taking the route used before the concept of MSJ was born, because joinings for this group have to be studied in the same spirit as that of the study of the centralizer and the factors.)

Del Junco and Rudolph generalized the notion of MSJ in [JRu] so that the theory of simplicity in a more general group includes Veech's results on prime transformations. We define a group G action on a measure space X to be graphic if the only ergodic self-joinings are the product measure and the measures sitting on graphs $\{(\varphi x, \phi x) : x \in X,\ \varphi, \phi \in \mathscr{L}(G)\}$. A simple action defined in [JRu] can be considered a special case of graphic actions because if every $\varphi \in \mathscr{L}(G)$ is invertible, then a graphic action is a simple action. We show in §3 that H action on T^2 is graphic. How many of the properties of simple actions in [JRu] can be extended to graphic actions remains to be investigated. We will provide an example of the type sought after in [JRu], that is, an example whose self-joinings consist of the product measure and the diagonal measure.

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1. The centralizer of H

It is clear that $\{mI : m \in Z\}$, where I denotes the identity matrix, is a subset of $\mathscr{L}(H)$. We will show that $\mathscr{L}(G)$ consists only of these elements. Given y_0, we define a horizontal line U_{y_0} to be the set $\{(x, y_0) : 0 \le x \le 1\} \subset T^2$. A vertical line V_{x_0} is analogously defined. The thrust of the argument hinges on the observation that two matrices $p = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $q = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ are rotations on each horizontal and vertical line, respectively. We note that

$$p^m \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + my \\ y \end{pmatrix} \pmod 1 \quad \text{and} \quad q^m \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + mx \end{pmatrix} \pmod 1.$$

By a (measurable) horizontal strip, we mean a set $\{(x, y) : 0 \le x < 1,\ y \in E$ for some measurable subset $E \subset S^1\}$. A (measurable) vertical strip is defined likewise. We will denote a circle S^1 by S hereafter.

It is easy to check that if ψ is an invertible measure-preserving transformation and ϕ commutes with ψ, then $\phi^{-1}A$ is an invariant subset of ψ for any

invariant subset A of ψ. It is also easy to see that any horizontal and vertical strip is invariant under the actions p and q, respectively.

Lemma 1.1. *If A is an invariant subset of positive measure under the action p, then A is a horizontal strip.*

Proof. If A is measurable, then $A \cap U_y$ is measurable for almost every y. Assume $\mu(A \cap U_{y_0}) \geq 0$ for some irrational y_0. It is enough to show $\mu(A \cap U_{y_0}) = 1$. Since $p = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is a rotation by y_0 on $A \cap U_{y_0}$, which is ergodic for any irrational y_0, the invariant subset $A \cap U_{y_0}$ of positive measure has to have full measure.

Corollary 1.2. *Any invariant subset of positive measure under q is a vertical strip.*

Let ϕ be an element in $\mathscr{L}(H)$.

Lemma 1.3. *If A is a horizontal (vertical) strip, then so is $\phi^{-1}(A)$.*

Proof. This is clear from Lemma 1.1 because $\phi^{-1}(A)$ is an invariant subset under p (q).

Lemma 1.4. *ϕ maps a horizontal line to a horizontal line.*

Proof. It is enough to show that $\phi^{-1}(U_y)$ is a union of horizontal lines for almost every y. Let y_0 be given. We define $A^\varepsilon = \{(x, y) : 0 \leq x < 1, y_0 - \varepsilon < y < y_0 + \varepsilon\}$. If we let $\varepsilon_i \to 0$, then we have $\phi^{-1}(U_{y_0}) = \phi^{-1}(\bigcap_i A^{\varepsilon_i}) = \bigcap_i \phi^{-1}(A^{\varepsilon_i})$. Since $\{\phi^{-1}(A^{\varepsilon_i})\}$ is a decreasing sequence of horizontal strips, $\phi^{-1}(U_{y_0})$ is a horizontal strip of measure zero.

Corollary 1.5. *ϕ maps a vertical line into a vertical line.*

Remark. It is not as yet clear that $\phi(U_y)$ is measurable in U_{y_0} or that $\phi(U_y)$ has full measure in U_{y_0}.

By Lemma 1.4 and Corollary 1.5, $P(x) = \pi_1\phi(V_x)$ and $Q(y) = \pi_2\phi(U_y)$ are well-defined functions on S, where π_1 and π_2 are the projections to the x- and y-coordinates, respectively.

Proposition 1.6. *If ϕ commutes with both p and q, then ϕ is of the type $P \times Q$ on T^2. That is, $\phi\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} Px \\ Qy \end{smallmatrix}\right)$.*

Proof. This is clear from Lemma 1.4 and Corollary 1.5 because $\phi\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is determined by $\phi(V_x)$ and $\phi(U_y)$.

Theorem 1.7. *If $\phi \in \mathscr{L}(H)$, then $\phi = mI$ for some $m \in Z$.*

Proof. Let $r = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of H. Since

$$r\phi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} P(x) \\ Q(y) \end{pmatrix} = \begin{pmatrix} aP(x) + bQ(y) \\ cP(x) + dQ(y) \end{pmatrix}$$

and

$$\phi r\begin{pmatrix} x \\ y \end{pmatrix} = \phi\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} P(ax + by) \\ Q(cx + dy) \end{pmatrix},$$

we have

$$\begin{pmatrix} aP(x) + bQ(y) \\ cP(x) + dQ(y) \end{pmatrix} = \begin{pmatrix} P(ax + by) \\ Q(cx + dy) \end{pmatrix}.$$

If we take $r = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ we have

$$\begin{pmatrix} P(x) + mQ(y) \\ Q(y) \end{pmatrix} = \begin{pmatrix} P(x+my) \\ Q(y) \end{pmatrix}. \tag{1}$$

If we take $r = \left(\begin{smallmatrix} 1 & 0 \\ m & 1 \end{smallmatrix}\right)$, we have

$$\begin{pmatrix} P(x) \\ mP(x) + Q(y) \end{pmatrix} = \begin{pmatrix} P(x) \\ Q(mx+y) \end{pmatrix}. \tag{2}$$

If we take $m = 1$ for (1) and (2), we have

$$P(x) + Q(y) = P(x+y) = Q(x+y) \quad \text{for almost every } \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3}$$

Fix x_0 such that for almost every $y \in S$, we have (3). We have $P(t) = Q(t)$ for almost every $t \in S$. Hence ϕ can be written as $P \times P$. According to (3),

$$P(x) + P(y) = P(x+y) \quad \text{for almost every } \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4}$$

We may assume that property (4) holds for every $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ (see [Z]). Since any measurable homomorphism for a locally compact second countable group is continuous, P is a continuous function on S [Z]. Hence $P(x) = \alpha x$ for some α. Since $P(0) = P(1) = 0 \pmod 1$, α is an integer m. Hence $\phi\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} mx \\ my \end{smallmatrix}\right)$, which implies that ϕ is of the type $\left(\begin{smallmatrix} m & 0 \\ 0 & m \end{smallmatrix}\right)$.

2. Factors of H on T^2

We make use of two well-known results about an ergodic rotation on a circle: Any measure-preserving transformation that commutes with the rotation by α (denoted by R_α) on a circle is another rotation. If R_α is isomorphic to R_β, then $\alpha = \beta$ or $-\beta$, and the isomorphism is via another rotation or a flip which maps x to $1 - x$, respectively.

We say a factor $(X, G, \mathscr{F}_0, \mu)$ of $(X, G, \mathscr{F}, \mu)$ has a finite fiber if a factor map $\varphi\colon (X, G, \mathscr{F}, \mu) \to (X, G, \mathscr{F}_0, \mu)$ is finite to one. When there is no possibility of confusion, we will sometimes call $\mathscr{F}_0$ a factor.

Lemma 2.1. *Let R_α be an ergodic rotation on a circle. Then any factor has a finite fiber.*

Proof. Since R_α is a simple action [V, JRu], each factor corresponds to a compact subgroup of $\mathscr{L}(R_\alpha) = \{R_\beta : \beta \in [0, 1)\}$ in the sense of [JRu]. That is, each factor gives rise to a compact subgroup (of the centralizer) whose element fixes every element of the factor $\mathscr{F}_0$. It is clear that each compact subgroup is generated by $R_{1/n}$ for some integer n. Let $\{R_{k/n_0} : k = 1, \dots, n_0\}$ be the compact subgroup of $\mathscr{L}(R_\alpha)$ corresponding to $\mathscr{F}_0$. If $A \in \mathscr{F}_0$, then A is invariant under rotation by $1/n_0$. That is, if $x \in A$, then $x + k/n_0 \in A$ for $i = 1, \dots, n_0$. Thus the factor identifies all the points $x + k/n_0$ for $i = 1, \dots, n_0$. If $y \neq x + k/n_0$ for any $k = 1, \dots, n_0$, then clearly x and y are different points in $\mathscr{F}_0$. Hence our lemma follows.

Proposition 2.2. *If $(T^2, H, \mathscr{F}_0, \nu)$ is a factor of $(T^2, H, \mathscr{F}, v)$, then $(T^2, H, \mathscr{F}_0, \nu)$ has a finite fiber. Moreover, each horizontal line and vertical line has the same number of points identified by the factor.*

Proof. We note that $\mathscr{F}_0$ is invariant under the actions p and q. Hence, by Lemma 2.1, there are finitely many points identified by $\mathscr{F}_0$ for almost every horizontal and vertical line. Let $h(x)$ denote the number of points identified in V_x and $h(y)$ denote the number of points identified in U_y. We want to show $h(x) = h(y)$, independent of x and y.

Let $A_l = \{(x, y) : h(y) = l\}$ and $B_k = \{(x, y) : h(x) = k\}$. Note that A_l and B_k are horizontal and vertical strips respectively. There exist l_0 and k_0 such that $\nu(A_{l_0}) > 0$ and $\nu(B_{k_0}) > 0$. Let (x_0, y_0) be a point in $A_{lo} \cap B_{ko}$. We let $k' = h(x_0 + 1/l_0)$. We consider the action q on V_{x_o} and V_{x_o+1/l_o}. For all m, q^m acts on V_{x_o} as a rotation by mx_0 and on V_{x_o+1/l_o} as a rotation by $m(x_0 + 1/l_0)$. We note that q^m acts on the factor spaces of V_{x_o} and V_{x_o+1/l_o} as a rotation by mk_0x_0 and $mk'(x_0 + 1/l_0)$ (mod 1) respectively. Hence we have

$$mk_0x_0 = mk'(x_0 + 1/l_0) \pmod 1 \quad \text{for all } m,$$
$$mx_0(k_0 - k') = mk'/l_0 \pmod 1 \quad \text{for all } m.$$

If we choose x_0 to be irrational, the above equalities hold if and only if $k_0 = k'$ and mk' is a multiple of l_0 for all m. Hence we have $k_0 = k'$, and k_0 is a multiple of l_0.

Likewise, when we consider U_{y_o} and $U_{y_0+1/k}$, we have $l_0 = l'$, where $l' = h(y_0 + 1/k_0)$ and l_0 is a multiple of k_0. Therefore we have $k' = l' = k_0 = l_0$. Since this is true for any pair (k_0, l_0) with $\nu(A_{l_o}) > 0$ and $\nu(B_{k_o}) > 0$, it is easy to see that all k's are the same as l_0 by fixing the set A_{l_o} and varying the set B_k of positive measure. Hence each horizontal and vertical line has the same number of points identified by $\mathscr{F}_0$. That is, $\mathscr{F}_0$ identifies all the points $\{(x + i/l_0, y + j/l_0)$, where $i, j = 1, \dots, l_0\}$.

Let (x', y') be a point identified with (x_0, y_0) by $\mathscr{F}_0$. Since p acts on the factor space of $U_{y'}$ as a rotation by l_0y' and on the factor space of U_{yo} as a rotation by l_0y_0, we have

$$ml_0y' = ml_0y_0 \pmod 1 \quad \text{for all } m,$$
$$ml_0(y' - y_0) = 0 \pmod 1 \quad \text{for all } m.$$

Hence $y' - y_0$ is a multiple of $1/l_0$. Likewise, $x' - x_0$ is a multiple of $1/l_0$. Therefore there are only l_0^2 points identified by $\mathscr{F}_0$.

Theorem 2.3. *There is a one-to-one correspondence between the centralizer and the collection of factors.*

Proof. Consider a matrix $\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)$ acting on $(T^2, H, \mathscr{F}, \nu)$. This map gives rise to a factor identifying all the points $(x+i/k, y+j/k)$ for $i, j = 1, 2, \dots, k$. Since all factors are of this type by Proposition 2.2, our proof is complete.

3. Self-joinings

The concept of minimal self-joinings was introduced by Rudolph to provide a machinery for constructing (counter) examples of certain properties. When we prove the property of MSJ in specific examples of Z-, R-, or Z^2-actions, we use the mixing property or the pointwise ergodic theorem. Since we do not have the pointwise ergodic theorem for the group under consideration, we use the ideas in §§1 and 2 to study its self-joinings.

We remark that the only joining of two rotations (S, R_α, μ) and (S, R_β, μ), where α and β are irrationally related (α and β are irrationals) is the product

measure. (By a joining of two rotations, we mean an ergodic invariant measure under $R_\alpha \times R_\beta$ on $S \times S$ with the right marginals.) If $\beta = (l/k)\alpha + (l'/k)$, where l and k are integers and are relatively prime, then the joinings are translates of the relative simple joining. That is, the joinings are the measures sitting on the graphs $\{(lx+\tau, kx) : x \in S \text{ for some } \tau \in [0, 1)\}$ [JRu]. We call τ the length of the translate.

We also note that by its definition a joining λ is invariant (not necessarily ergodic) under $p \times p$ and $q \times q$. Consider $T^2 \times T^2 = S_1 \times S_2 \times S_3 \times S_4 = \{(x, y, z, w) : (x, y) \in T^2, (z, w) \in T^2\}$. We denote λ restricted to the first (second) and the third (fourth) coordinates by $\lambda_{1,3}$ $(\lambda_{2,4})$. If A is a subset of S_1, then $\overline{A}$ denotes the set $\{(x, y) : x \in A, 0 \le y < 1\}$. If B is a subset of S_3, then $\overline{B}$ is defined likewise. By the definition of $\lambda_{1,3}$, $\lambda_{1,3}(A \times B) = \lambda(\overline{A} \times \overline{B})$. We denote the conditional measure λ on the set $\{x\} \times S_2 \times \{z\} \times S_4$ by $\lambda_{x,z}$. We define $\lambda_{y,w}$ likewise. We note that $\lambda = \int \lambda_{x,z}\, d\lambda_{1,3}$, where the integral takes the usual meaning, that is, $\lambda(C) = \int \lambda_{x,z}(C)\, d\lambda_{1,3}$ for any measurable subset C of $T^2 \times T^2$. Given x and z, $q \times q$ acts on $\{x\} \times S_2 \times \{z\} \times S_4$ as a rotation by x on S_2 and by z on S_4. Specifically, $q \times q(x, y, z, w) = (x, y+x, z, w+z)$ for $y \in S_2$ and $w \in S_4$.

Lemma 3.1. *Let $\lambda_{x,z}$ be a nontrivial measure. Then $\lambda_{x,z}$ is invariant under $q \times q$ for almost every (x, z) with respect to $\lambda_{1,3}$.*

Proof. Suppose not. Let $L = \{(x, z) : \lambda_{x,z} \text{ is not invariant}\}$ be the set of positive measure with respect to $\lambda_{1,3}$. For each $(x, z) \in L$, there exists a number $\rho(x, z) < 1$ and a subset $N_{\rho(x,z)}$ of $\{x\} \times S^1 \times \{z\} \times S^1$ such that

$$\lambda_{x,z}((q \times q)N_{\rho(x,z)}) = \rho(x, z)\lambda_{x,z}(N_{\rho(x,z)}) > 0.$$

There exists $\gamma < 1$ such that $\lambda(N_\gamma) > 0$, where

$$N_\gamma = \bigcup_{(x,z)\in L} \{N_{\rho(x,z)} : \rho(x, z) < \gamma\}.$$

We have

$$\begin{aligned}\lambda(q \times q(N_\gamma)) &= \int_{\{(x,z)\in L\,:\,\rho(x,z)<\gamma\}} \lambda_{x,z}((q \times q)N_{\rho(x,z)})\, d\lambda_{1,3} \\ &< \rho(x, z) \int_{\{(x,z)\in L\,:\,\rho(x,z)<\gamma\}} \lambda_{x,z}(N_{\rho(x,z)})\, d\lambda_{1,3} \\ &< \gamma\lambda(N_\gamma).\end{aligned}$$

Since λ is invariant under $q \times q$, this is a contradiction.

Lemma 3.2. *$\lambda_{x,z}$ is composed of either $\mu \times \mu$ or translates of the relative, simple joining.*

Proof. Since $\lambda_{x,z}$ is invariant under $q \times q$, the marginals of $\lambda_{x,z}$ are invariant under $q \times q$. That is, $\lambda_{x,z}(A_2 \times S_4) = \lambda_{x,z}((q \times q)(A_2 \times S_4)) = \lambda_{x,z}(qA_2 \times S_4)$, where qA_2 has the obvious meaning. If we define $\lambda^1_{x,z}(A_2) = \lambda_{x,z}(A_2 \times S_4)$ and $\lambda^2_{x,z}(A_4) = \lambda_{x,z}(S_2 \times A_4)$, then $\lambda^1_{x,z}$ is invariant under the rotation by x and $\lambda^2_{x,z}$ is invariant under the rotation by z. If x and z are irrationals, then $\lambda^1_{x,z}$ and $\lambda^2_{x,z}$ are both Lebesgue measure. (An irrational rotation on a circle is uniquely ergodic.) Hence, if x and z are irrationally related, then $\lambda_{x,z}$

having the right marginals is $\mu \times \mu$. Otherwise, it is composed of translates of the relative simple joining.

Corollary 3.3. *$\lambda_{y,w}$ is either $\mu \times \mu$ or a measure composed of translates of the relative simple joining.*

Proof. Since $\lambda_{y,w}$ is invariant under $p \times p$, the corollary follows as in Lemma 3.2.

Let r be the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in H.

Lemma 3.4. *If $\lambda_{x,z}$ is a product measure for almost every (x, z) with respect to $\lambda_{1,3}$, then λ is the product measure $\nu \times \nu$.*

Proof. Let $\overline{A} = \{(x, y) : 0 \le y < 1,\ x \in A$ for a measurable subset $A \subset S_1\}$ and $\overline{B} = \{(z, w) : 0 \le w < 1,\ z \in B$ for a measurable subset $B \subset S_3\}$. We denote $r(\overline{A})$ by $\overline{A'}$ and $r(\overline{B})$ by $\overline{B'}$. We have

$$\lambda_{1,3}(A \times B) = \lambda(\overline{A} \times \overline{B}) = \lambda(\overline{A'} \times \overline{B'}) = \int \lambda_{x,z}(\overline{A'} \times \overline{B'})\, d\lambda_{1,3} = \int \mu A \mu B\, d\lambda_{1,3}$$
$$= \mu A \mu B \int d\lambda_{1,3} = \mu A \mu B.$$

Hence $\lambda_{1,3}$ is the product measure, which in turn implies that λ is the product measure.

Theorem 3.6. *If there exists a subset of positive measure C with respect to $\lambda_{1,3}$ such that $\lambda_{x,z}$ is not a product measure for $(x, z) \in C$, then λ is a measure sitting on the graph $\{(Ku, Mu) : u \in T^2,\ K,\ M \in \mathscr{L}(H)\}$.*

Proof. If $\lambda_{x,z}$ is not a product measure, then x and z are rationally related, say $x = (k/m)z + (k'/m)$. Hence there exists k_0, k_0', and m_0 such that the subset $C_0 = \{(x, z) \in C : x = (k_0/m_0)z + (k_0'/m_0)\}$ has positive measure with respect to $\lambda_{1,3}$. For notational convenience, we write k/m instead of k_0/m_0. Let B denote a subset such that the set $\Lambda = \{(x, y, z, w) : x = (k/m)z + (k'/m),\ y \in S_2,\ z \in B,\ w \in S_4\}$ has positive measure. When we denote by Λ' the set $(r \times r)\Lambda = \{(x, y, z, w) : x \in S_1,\ y \in (k/m)w + (k'/m),\ z \in S_3,\ w \in B\}$, then we have $\lambda(\Lambda') = \lambda(\Lambda) > 0$. Therefore the subset $D = \{(x, z) : \lambda_{x,z}(\Lambda') > 0\}$ has positive measure with respect to $\lambda_{1,3}$. We note that $\lambda_{x,z}(\Lambda') > 0$ if and only if $x = (k/m)z \pmod{1/m}$. If $\lambda_{x,z}(\Lambda') > 0$, then $\lambda_{x,z}$ has the measure sitting on the graph $\{(kw, mw) : w \in S_4\}$ as one of its components.

Let Γ denote the set $\{(x, y, z, w) : (x, z) \in D,\ y = (k/m)w \pmod{1/m},\ w \in S_4\}$. We denote by Γ' the set $(r \times r)\Gamma = \{(x, y, z, w) : x = (k/m)z \pmod{1/m},\ z \in S_3,\ (y, w) \in D\}$. Let A be a subset of positive measure and k' be a nonnegative integer $< m$. We claim that the subset $\Gamma_{A,k'} = \{(x, y, z, w) : x = (k/m)z + (k'/m),\ z \in A,\ (y, w) \in D\}$ of Γ' has positive measure. We see this as follows: Suppose not. That is, there exist a subset A and a k' such that $\lambda(\Gamma_{A,k'}) = \lambda((r \times r)\Gamma_{A,k'}) = \lambda(\{(x, y, z, w) : (x, z) \in D,\ y = (k/m)w + (k'/m),\ w \in A\}) = 0$. This contradicts the fact that $\lambda_{x,z}$ for $(x, z) \in D$ has the relative simple joining as one of its components.

In particular, we have $\lambda_{1,3}(\{(x, z) : x = (k/m)z + (k'/m),\ z \in A\}) > 0$ and $\lambda_{x,z}(\Gamma') > 0$ for each (x, z) where $x = (k/m)z \pmod{1/m}$. If $x = (k/m)z$

$(\bmod 1/m)$, then $\lambda_{x,z}$ has the relative simple joining as one of its components. Therefore λ has the measure sitting on the graph $\{(kz, kw, mz, mw) : (z, w) \in S_3 \times S_4\} = \{(Ku, Mu) : u \in T^2, K, M \in \mathscr{L}(H)\}$ as one of its components. This is enough to determine the ergodic measure λ.

Corollary 3.7. *$(T^2, H, \mathscr{F}, \nu)$ is graphic.*

4. An example whose self-joinings consist of the product measure and the diagonal measure

It is well known that certain horocycle flows on a surface of constant negative curvature have the property of minimal self-joining but not minimal rescale joining [R]. This is so because a horocycle flow S^t is isomorphic to $S^{\alpha t}$ for all $\alpha \in R - \{0\}$. (Since $S^{\alpha t}$ also has MSJ, $S^{\alpha t}$ and S^t are disjoint in the sense of [F] if and only if they are not isomorphic.)

We denote the time-1 discrete map of the horocycle flow by ψ. Since the flow S^t has the property of MSJ, we know the centralizer of the action is $\{S^s : s \in R\}$. Since Z is a cocompact subgroup of R, ψ has the property of simplicity and $\mathscr{L}(\psi) = \mathscr{L}(S^t) = \{S^s : s \in R\}$ by [JRu]. Hence the set of self-joinings of ψ consists of the product measure and off-diagonal measures concentrated on $\{(x, S^s x)\}$ for some $s \in R$. We denote the off-diagonal measure by ν_s. Let φ be an isomorphism (via the geodesic flow) between S^{3t} and S^t. Consider the group G generated by $\{\psi, \varphi\}$ where $\varphi\psi^3 = \psi\varphi$. (We note that the group G is discrete; hence it is a unimodular group.)

Theorem 4.1. *The only self-joinings of G are $\mu \times \mu$ and $\nu_0 = \mu_\Delta$.*

Proof. Let λ be an invariant measure under $G \times G$, hence under $\psi \times \psi$. It is sufficient to show that λ is composed of $\mu \times \mu$ and $\nu_0 = \mu_\Delta$. The measure λ can be written as $\pi(\mu \times \mu) + (1 - \pi) \int \nu_t \, d\sigma(t)$, where σ is the unique Borel measure on R corresponding to λ. We denote $(1 - \pi)^{-1}(\lambda - \pi(\mu \times \mu))$ by λ_0. Clearly, λ_0 is an invariant measure. In particular, when $C = A \times B$,

$$\lambda_0(A \times B) = \int \nu_t(A \times B) \, d\sigma(t) = \int \mu(A \cap S^{-t}B) \, d\sigma(t).$$

Since λ_0 is invariant under $\varphi \times \varphi$, we have

$$\begin{aligned}
\lambda_0(A \times B) = \lambda_0(\varphi^{-1} \times \varphi^{-1}(A \times B)) &= \int \nu_t(\varphi^{-1}(A) \times \varphi^{-1}(B)) \, d\sigma(t) \\
&= \int \mu(\varphi^{-1}(A) \cap S^{-t}\varphi^{-1}(B)) \, d\sigma(t) \\
&= \int \mu(\varphi^{-1}(A) \cap \varphi^{-1}S^{-t/3}(B)) \, d\sigma(t) \\
&= \int \mu(\varphi^{-1}(A \cap S^{-t/3}(B))) \, d\sigma(t) = \int \mu(A \cap S^{-t/3}(B)) \, d\sigma(t).
\end{aligned}$$

The proof follows if we use the following lemma:

Lemma 4.2. *Let σ be a Borel probability measure on R satisfying*

$$\int \mu(A \cap S^{-t}B) \, d\sigma(t) = \int \mu(A \cap S^{-t/3}B) \, d\sigma(t),$$

for any subsets A and B. Then σ is the point measure at $t = 0$.

Proof. Let ε be given, and let s be any given positive real number. Choose t_0 $(> 3s)$ sufficiently large so that $\sigma([-t_0, t_0]) > 1 - \varepsilon$. Using a flow Rochlin tower, it is easy to find a subset A such that

$$\mu(A \cap S^{-t}A) = \begin{cases} \mu(A)(1 - |t|/s) & \text{for } |t| \le s, \\ 0 & \text{for } s \le |t| \le t_0. \end{cases}$$

$$\left| \int \mu(A \cap S^{-t}A)\, d\sigma(t) - \int_{-t_0}^{t_0} \mu(A \cap S^{-t}A)\, d\sigma(t) \right| = \int_{|t| \ge t_0} \mu(A \cap S^{-t}A)\, d\sigma(t) < \varepsilon\mu(A). \tag{4.1}$$

By our choice of A, we have

$$\int_{-t_0}^{t_0} \mu(A \cap S^{-t}A)\, d\sigma(t) = 2\int_0^s \mu(A)(1 - t/s)\, d\sigma(t).$$

Hence by our condition on σ and (4.1), we have

$$\left| \int_{-t_0}^{t_0} \mu(A \cap S^{-t/3}A)\, d\sigma(t) - \int_{-t_0}^{t_0} \mu(A \cap S^{-t}A)\, d\sigma(t) \right| < 2\varepsilon\mu(A).$$

Since t_0 is chosen to be bigger than $3s$, we have

$$2\mu(A)\left(\int_c^{3s} (1 - t/3s)\, d\sigma(t) - \int_c^s (1 - t/s)\, d\sigma(t) \right) > (4/3)\mu(A) \int (t/s)\, d\sigma(t).$$

Hence we have $(2/3)\int (t/s)\, d\sigma(t) < \varepsilon$. Since ε is arbitrary, this implies that σ is the point measure at $t = 0$.

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