ON THE MEAN CURVATURE ESTIMATES FOR BOUNDED SUBMANIFOLDS

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(Communicated by Jonathan M. Rosenberg)

Abstract. A Liouville-type theorem is proved for strongly subharmonic functions on complete Riemannian manifolds of bounded curvature. We use this to give a simple proof of a theorem of Jorge, Koutroufiotis and Xavier, which gives an estimate for the exterior size of a submanifold in terms of the sup of the length of its mean curvature.

We give a short proof of the following theorem of Jorge and Xavier [3].

Theorem 1. Let \( M \) and \( \overline{M} \) be Riemannian manifolds and let \( f : M \to \overline{M} \) be an isometric immersion. Suppose that \( M \) is complete with \( \inf \) scalar curvature \( > -\infty \). Let \( \delta := \sup \) of the sectional curvature of \( \overline{M} \), \( H := \sup \) of the length of mean curvature vector of \( f \), and \( \lambda \) be such that there exists some closed normal ball \( B_{\lambda}^{-} \) of radius \( \lambda \) in \( \overline{M} \) containing \( f(M) \). Then,

\[
\lambda \geq \begin{cases} 
\left(1/\sqrt{-\delta}\right) \tanh^{-1}\left(\sqrt{-\delta}/H\right) & \text{if } \delta < 0, \\
1/H & \text{if } \delta = 0, \\
\min\{(1/\sqrt{\delta}) \tan^{-1}(\sqrt{\delta}/H), \pi/(2\sqrt{\delta})\} & \text{if } \delta > 0.
\end{cases}
\]

Theorem 1 follows from

Theorem 2. Let \( M \) be a complete Riemannian manifold with bounded sectional curvature and \( \theta \) a positive constant. Then, every \( C^2 \) solution to the inequality \( \Delta u \geq \theta \) is unbounded.

Proof. First, we claim that given any \( r > 0 \), there exists \( \alpha > 0 \) such that for any \( p \in M \), we can construct a \( C^2 \) function \( \tilde{v}_{p,r} : M \to \mathbb{R}_+ \) such that \( \tilde{v}_{p,r}(p) = 1 \), \( \tilde{v}_{p,r} \) vanishes outside \( B_r(p) \), and \( |\text{Hess}_x \tilde{v}_{p,r}| \leq \alpha \) for all \( x \in M \). That such a \( \tilde{v}_{p,r} \) exists independent of the injectivity radius of \( M \) follows by smoothing a bump function by convolution in the tangent space using the techniques of Theorem 1.8 in [1]. Set \( v_{p,r} := \tilde{v}_{p,r}/(\alpha \sqrt{d}) \), where \( d := \dim M \). Then, \( |\text{Hess} v_{p,r}| \leq \theta/\sqrt{d} \).

Now, assume that \( u \) is bounded. Let \( n := \sup u \). Then for \( r > 0 \) fixed, there exists a point \( q \in M \) such that \( n - u(q) < \theta/(\alpha \sqrt{d}) \) where \( \alpha \) is as in the

Received by the editors October 22, 1989 and, in revised form, November 14, 1989 and July 26, 1990.

1980 Mathematics Subject Classification (1985 Revision). Primary 53C20; Secondary 53C42.

The work was partially supported by NSF EPSCoR grant RII-8610669.

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construction above. Then,

$$\Delta (u + v_{q,r}) \geq \Delta u - |\Delta v_{q,r}| \geq \Delta u - \frac{|\text{Hess } v_{q,r}|}{\sqrt{d}} \geq \theta - \theta = 0,$$

so the function $u + v_{q,r}$ is subharmonic on $M$. On the other hand, $u(q) + v_{q,r}(q) = n - \theta/(\alpha \sqrt{d}) + \theta v_{q,r}(q)/(\alpha \sqrt{d}) = n$ while $u + v_{q,r}$ coincides with $u$ outside $B_r(q)$. Therefore, $u + v_{q,r}$ must attain some maximum point in the set $B_r(q)$, contradicting the maximum principle.

**Proof of Theorem 1.** By the assumption on the scalar curvature and the Gauss equation, the sectional curvature of $M$ is bounded (cf. [2, p. 722]). Assume that there exists a point $o \in M$ such that $f(M) \subset B_o(o)$. If $\delta > 0$, assume also that $\lambda < \pi/(2\sqrt{\delta})$. Then, it is well known (cf. [2, 3]) that the function $\varphi : M \to \mathbb{R}_+$ defined by $\varphi(x) := \text{dist}_M(x, o)^2$ satisfies $\Delta \varphi(x) \geq d(e - H\lambda) \lambda \sqrt{\delta} \coth(\lambda \sqrt{\delta})$ if $\delta < 0$,

$$\varepsilon := \begin{cases} \lambda \sqrt{-\delta} \coth(\lambda \sqrt{-\delta}) & \text{if } \delta < 0, \\
1 & \text{if } \delta = 0, \\
\lambda \sqrt{\delta} \cot(\lambda \sqrt{\delta}) & \text{if } \delta > 0. \end{cases}$$

Hence substituting, if $\lambda$ is less than the prescribed constant, $\Delta \varphi \geq \varepsilon := \sqrt{d} \varepsilon$. By Theorem 1, $\varphi$ is unbounded, which is absurd.

**References**


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