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THOM MODULES AND mod p SPHERICAL FIBRATIONS

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ABSTRACT. In this paper we show that every finite Thom module over the ring of invariants of a finite nonmodular group can be realized as $\bmod p$ cohomology of the Thom space of a spherical fibration.

INTRODUCTION

The concept of Thom module was first defined by D. Handel [3]. We recall here the definition of Thom module in the particular context of unstable algebras over the Hopf algebra of Steenrod reduced pth-powers $\mathscr{A}_p^*$ as given in [2]:

Definition. Let R^* be an unstable algebra over $\mathscr{A}_p^*$. An R^*-Thom module, M^*, is both an $\mathscr{A}_p^*$-module and a free R^*-module of rank one such that for any $\theta \in \mathscr{A}_p^*$ with $\Delta(\theta) = \sum_i \theta_i' \otimes \theta_i''$ and $r \in R^*$, $m \in M^*$, we have the relation $\theta(r \cdot m) = \sum_i \theta_i'(r) \cdot \theta_i''(m)$. A homogeneous generator of M^*, $U \in M^*$, as an R^*-module is called a Thom class for M^*.

The motivating examples are the reduced $\bmod p$ cohomology groups of Thom complexes. In fact, if $\xi \downarrow X$ is an oriented vector bundle and $T(\xi \downarrow X)$ its Thom complex, $\tilde{H}^*(T(\xi \downarrow X); \mathbf{F}_p)$ is an $H^*(X; \mathbf{F}_p)$-Thom module.

We can generalize this example to the case of spherical fibrations: A $\bmod p$ spherical fibration is an orientable Hurewicz fibration $E \to X$ whose fiber has the homotopy type of a p-complete sphere. The associated Thom space $T(E \to X)$ is the homotopy cofiber of $E \to X$. Its reduced $\bmod p$ cohomology is also an example of an $H^*(X; \mathbf{F}_p)$-Thom module.

Definition. The characteristic classes of an R^*-Thom module M^* are $q(M^*) = 1 + q_1(M^*) + q_2(M^*) + \cdots$, $q_i(M^*) \in R^*$ such that $\mathscr{P}^i(U) = q_i(M^*) \cdot U$.

The characteristic classes of M^* determine the $\mathscr{A}_p^*$-structure of M^* and then its isomorphism class. A finite Thom module is a Thom module for which all characteristic classes, except possibly a finite number of them, vanish. The examples above are clearly finite Thom modules.

Let V_n denote the $\mathbf{F}_p$ vector space $H^2(BT^n; \mathbf{F}_p)$, then the symmetric algebra $P_n = S(V_n)$ is $H^*(BT^n; \mathbf{F}_p) \cong \mathbf{F}_p[t_1, \dots, t_n]$. If $G \leq GL_n(\mathbf{F}_p)$, G acts on

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P_n by linear transformations and the ring of invariants P_n^G inherits an unstable $\mathscr{A}_p^*$-action. In [2] it is shown that any finite P_n^G-Thom module is isomorphic to one of the form $P_n^G \cdot U$, where $U \in P_n$ is a product of pre-Euler classes (which are defined below). Recall from [5] that a pre-Euler class of a nontrivial orbit, $B \subset V_n$, by the action of G is the product of any collection of elements of B which is maximal with respect to the property that its elements be pairwise linearly independent.

When G is nonmodular—that is, the order of G is prime to p—the invariant algebras P_n^G are realizable as $\bmod p$ cohomology algebras of spaces (we give a model below) so it makes sense to ask whether any finite P_n^G-Thom module is realizable as the reduced $\bmod p$ cohomology of a Thom space. In [2] it is shown that since an orientable vector bundle has Pontrjagin classes, some finite P_n^G-Thom modules cannot appear as reduced $\bmod p$ cohomology of the Thom complex of an orientable vector bundle and the cases in which it is possible are classified.

In this note we want to show that if we consider Thom spaces of $\bmod p$ spherical fibrations, then any finite P_n^G-Thom module, with nonmodular G, is realizable, more precisely:

Theorem 1. *Let M^* be a P_n^G-finite Thom module, where G is a nonmodular subgroup of $GL_n(\mathbf{F}_p)$. There exists a space X with $H^*(X; \mathbf{F}_p) \cong P_n^G$ and a $\bmod p$ spherical fibration $E \to X$ such that*

$$\tilde{H}^*(T(E \to X); \mathbf{F}_p) \cong M^*$$

as P_n^G-Thom modules.

Remark. It is a consequence of the nonrealizability of general finite Thom modules as Thom spaces of vector bundles that the fibrations in the theorem do not arise, generally, as spherical fibrations associated to vector bundles.

Next we describe the model of a space with cohomology P_n^G. Let $\mathbf{Z}/p^\infty$ be the p-torsion subgroup of S^1. The induced map $B\mathbf{Z}/p^\infty \to BS^1$ is an isomorphism in $\bmod p$ cohomology and we identify $H^*((B\mathbf{Z}/p^\infty)^n; \mathbf{F}_p) \cong \mathbf{F}_p[t_1, \dots, t_n]$. Now, a p-adic number a represents an endomorphism of $\mathbf{Z}/p^\infty$: $\theta \mapsto \theta^a$. Therefore, given an $n \times n$ matrix over $\hat{\mathbf{Z}}_p$, $M = (a_{ij})$, we may define the homomorphism

$$\begin{aligned} x = (\theta_1, \dots, \theta_n) \in (\mathbf{Z}/p^\infty)^n &\mapsto xM \\ &= (\theta_1^{a_{11}} \cdots \theta_n^{a_{n1}}, \dots, \theta_1^{a_{1n}} \cdots \theta_n^{a_{nn}}) \in (\mathbf{Z}/p^\infty)^n \end{aligned}$$

and this one induces a self map of $(B\mathbf{Z}/p^\infty)^n$. Thus we obtain a right action of $GL_n(\hat{\mathbf{Z}}_p)$ on $(B\mathbf{Z}/p^\infty)^n$.

Assume G is a subgroup of $GL_n(\mathbf{F}_p)$ of order prime to p. This condition ensures the existence of a lifting $G \to GL_n(\hat{\mathbf{Z}}_p)$ and, hence, that G acts on $(B\mathbf{Z}/p^\infty)^n$ from the right. We recover the original $\bmod p$ representation of G as the induced left action of G on $\bmod p$ cohomology of $(B\mathbf{Z}/p^\infty)^n$. The fact that $|G| \not\equiv 0 \bmod p$ also implies that the map $(B\mathbf{Z}/p^\infty)^n \to (B\mathbf{Z}/p^\infty)^n \times_G EG$ induces an isomorphism of $H^*((B\mathbf{Z}/p^\infty)^n \times_G EG; \mathbf{F}_p)$ onto the invariant ring of $\mathbf{F}_p[t_1, \dots, t_n] \cong H^*((B\mathbf{Z}/p^\infty)^n; \mathbf{F}_p)$ by the action of G.

Finite Thom modules over $\mathbf{F}_p[t^{p-1}]$

Assume $\mathbf{Z}/p-1$ acts on $\mathbf{F}_p[t]$ in the obvious way. Any finite Thom module over $\mathbf{F}_p[t]^{Z/p-1} = \mathbf{F}_p[t^{p-1}]$ is isomorphic to one of the form $t^k\mathbf{F}_p[t^{p-1}]$, $k \geq 0$, whose characteristic classes are $q = (1+t^{p-1})^k$. According to [2] these Thom modules can only appear as the cohomology of the Thom complex of a vector bundle whenever $(p-1)/2$ divides k.

Suppose $X = B\mathbf{Z}/p^\infty \times_{\mathbf{Z}/p-1} E\mathbf{Z}/p-1$, so that $H^*(X; \mathbf{F}_p) \cong \mathbf{F}_p[t^{p-1}]$.

Proposition 2. *There are* mod p *spherical fibrations* $\hat{S}_p^{2k+1} \to E_k \to X$ *with characteristic classes* $q = (1+t^{p-1})^k$, *so that the corresponding Thom spaces realize the Thom modules* $t^k\mathbf{F}_p[t^{p-1}]$.

Proof. It is enough to show these fibrations exist for $k < p-1$ as we know that there is a complex bundle of dimension $(p-1)$ with q-classes $(1+t^{p-1})^{p-1}$ over $B\mathbf{Z}/p^\infty \times_{\mathbf{Z}/p-1} E\mathbf{Z}/p-1$ [2].

Assume $A_n = (\mathbf{F}_p^*)^n$ and $SA_n = \{(\theta_1, \dots, \theta_n) \in A_n \mid \theta_1\theta_2\cdots\theta_n = 1\}$ and define G_n and SG_n to be the extensions $A_n \tilde{\times} \Sigma_n$ and $SA_n \tilde{\times} \Sigma_n$, where the symmetric group Σ_n acts on A_n and SA_n permuting the components. We now consider the spaces

$$X_n = (B\mathbf{Z}/p^\infty)^n \times_{G_n} EG_n, \qquad Y_n = (B\mathbf{Z}/p^\infty)^n \times_{SG_n} ESG_n,$$

where we let G_n and SG_n act on $(B\mathbf{Z}/p^\infty)^n$ by representing $\mathbf{F}_p^*$ as $(p-1)$ roots of unit in $\hat{\mathbf{Z}}_p$. Provided $n < p$, the mod p cohomology rings of X_n and Y_n should be identified with the rings of invariants of $H^*((B\mathbf{Z}/p^\infty)^n; \mathbf{F}_p) \cong \mathbf{F}_p[t_1, \dots, t_n]$ by the action of G_n and SG_n respectively; that is, for $n < p$

$$H^*(X_n; \mathbf{F}_p) \cong \mathbf{F}_p[q_1, \dots, q_n],$$
$$H^*(Y_n; \mathbf{F}_p) \cong \mathbf{F}_p[q_1, \dots, q_{n-1}, E_n],$$

where $1 + q_1 + \cdots + q_n = \prod_{i=1}^n (1 + t_i^{p-1})$ and $E_n = t_1 \cdots t_n$.

Let $i_k: (B\mathbf{Z}/p^\infty)^k \hookrightarrow (B\mathbf{Z}/p^\infty)^{k+1}$ be the inclusion of the first k factors. The monomorphism $j_k: (\theta_1, \dots, \theta_k) \in G_k \mapsto (\theta_1, \dots, \theta_k, (\theta_1\cdots\theta_k)^{-1} \in SG_{k+1}$ makes i_k equivariant and so this induces

$$f_k: X_k \to Y_{k+1}.$$

It follows easily that under the above identifications the induced map in mod p cohomology is

$$\begin{aligned} f_k^*: H^*(Y_{k-1}; \mathbf{F}_p) &\to H^*(X_k; \mathbf{F}_p), \\ q_i &\mapsto q_i \quad \text{for } i \leq k, \\ E_{k+1} &\mapsto 0 \end{aligned}$$

provided $k < p-1$.

Let F_k be the homotopy fiber of the induced map between the respective p-completions of X_k and Y_{k+1}: $F_k \to (X_k)_p^\wedge \to (Y_{k+1})_p^\wedge$.

Lemma 3. *If* $k < p-1$, F_k *has the same homotopy type as* $\hat{S}_p^{2k+1}$.

Proof. Since fundamental groups of X_k and Y_{k-1} are p-perfect the respective p-completions are simply connected spaces (see [1]).

So, we may use the Eilenberg-Moore spectral sequence to calculate the $\bmod p$ cohomology of F_k. It follows that $H^*(F_k; \mathbf{F}_p) = E(x_{2k+1})$, that is, F_k has the same $\bmod p$ cohomology as S^{2k+1}. Now, since $H_1(F_p; \mathbf{F}_p) = 0$, F_k is a $\mathbf{Z}/p$-good space. Moreover, the p-completion functor preserves orientable fibrations, so F_k is p-complete. It follows that F_k has actually the homotopy type of $\hat{S}_p^{2k+1}$.

Next, we calculate the characteristic classes of the $\hat{S}_p^{2k+1}$-fibrations we have just found. Assume a fixed $k < p-1$. The Thom space of the constructed $\hat{S}_p^{2k+1}$-fibration is the cofiber of $\hat{f}_k$:

$$(X_k)_p^\wedge \xrightarrow{\hat{f}_k} (Y_{k+1})_p^\wedge \xrightarrow{j} T.$$

By the Thom isomorphism theorem $\tilde{H}^*(T; \mathbf{F}_p) \cong U \cdot H^*((Y_{k+1})_p^\wedge; \mathbf{F}_p) \cong U \cdot \mathbf{F}_p[q_1, \dots, q_k, E_{k+1}]$ with $\deg U = 2k+2$. Since $\hat{f}_k^*$ is onto, the cohomology exact sequence of the cofibration becomes the short exact sequence:

$$0 \to U \cdot \mathbf{F}_p[q_1, \dots, q_k, E_{k+1}] \xrightarrow{j^*} \mathbf{F}_p[q_1, \dots, q_k, E_{k+1}] \xrightarrow{\hat{f}_k^*} \mathbf{F}_p[q_1, \dots, q_k] \to 0.$$

It follows that the $\bmod p$ Euler class is $j^*(U) = E_{k+1}$. If $\mathscr{P} = 1 + \mathscr{P}^1 + \mathscr{P}^2 + \cdots$ denotes the total Steenrod power operation,

$$\mathscr{P}(E_{k+1}) = \left(\prod_{i=1}^{k+1}(1 + t_i^{p-1})\right)E_{k+1} = (1 + q_1 + q_2 + \cdots + q_{k+1})E_{k+1}$$

and then we also have $\mathscr{P}(U) = (1 + q_1 + q_2 + \cdots + q_{k+1})U$; that is, $1 + q_1 + q_2 + \cdots + q_{k+1} \in H^*((Y_{k+1})_p^\wedge; \mathbf{F}_p)$ are just the characteristic classes of our fibration.

Finally, if

$$g_k \colon X = B\mathbf{Z}/p^\infty \times_{\mathbf{Z}/p-1} E\mathbf{Z}/p-1 \to (Y_{k+1})_p^\wedge = ((B\mathbf{Z}/p^\infty)^{k+1} \times_{SG_{k+1}} ESG_{k+1})_p^\wedge$$

is the map induced by $x \in B\mathbf{Z}/p^\infty \mapsto (x, \dots, x, e) \in (B\mathbf{Z}/p^\infty)^{k+1}$, where e is the base point of $B\mathbf{Z}/p^\infty$, and the group monomorphism $\theta \in \mathbf{Z}/p-1 \mapsto (\theta, \dots, \theta, \theta^{-k}) \in SG_{k+1}$, then, the pull-back along g_k of our $\hat{S}_p^{2k+1}$-fibration is an orientable $\hat{S}_p^{2k+1}$-fibration, $E_k \to X$, with characteristic classes:

$$\begin{aligned} q(E_k \to X) &= g_k^*(1 + q_1, \dots, q_{k+1}) \\ &= g_k^*\left(\prod_{i=1}^{k+1}(1 + t_i^{p-1})\right) = (1 + t^{p-1})^k. \quad \square \end{aligned}$$

Proof of Theorem 1

Assume M^* is a finite P_n^G-Thom module, G a nonmodular subgroup of $GL_n(\mathbf{F}_p)$. From [2] we know that this module is isomorphic, as P_n^G-Thom module, to one of the form $P_n^G \cdot f$ with $f = \prod_{i=1}^k E_i^{m_i}$, a product of pre-Euler elements. If, for each i, $\zeta_i \downarrow X$ is a $\bmod p$ spherical fibration such that $\tilde{H}^*(T(\zeta_i \downarrow X); \mathbf{F}_p) \cong P_n^G \cdot E_i$, then it is clear that a fiberwise join procedure will give us a fibration $\zeta \downarrow X$ with $\tilde{H}^*(T(E \to X); \mathbf{F}_p) \cong M^*$.

It is then enough to prove the theorem in the case $M^* \cong P_n^G \cdot E[w]$, where $E[w]$ is the pre-Euler class associated to an orbit: $[w]$, $w \in V_n$, $w \neq 0$.

Suppose $E[w] = \prod_{i=1}^m w_i$ so that characteristic classes of the finite P_n^G-Thom module $P_n^G \cdot E[w]$ are $q = \prod_{i=1}^m (1 + w_i^{p-1})$. The theorem then follows from the next proposition:

Proposition 4. *There is a* mod p *spherical fibration* $\zeta \downarrow (B\mathbf{Z}/p^\infty)^n \times_G EG$ *with characteristic classes* $q(\zeta) = \prod_{i=1}^m (1 + w_i^{p-1})$.

Proof. Define

$$H = \{g \in G \mid gw = w\}$$
$$\overline{H} = \{g \in G \mid gw = \chi(g)w\,, \text{ for some } \chi(g) \in \mathbf{F}_p^*\}.$$

Then, H is a normal subgroup of $\overline{H}$ and $\overline{H}/H$ is a subgroup of $\mathrm{Aut}\langle w\rangle \cong \mathbf{F}_p^*$ which is in a one-to-one correspondence with $[w] \cap \langle w\rangle$. Moreover, if $G/\overline{H} = \{\tau_1\overline{H}, \tau_2\overline{H}, \ldots, \tau_m\overline{H}\}$, each $\tau_i w$ belongs to a different line and we may assume that $w_i = \tau_i w$.

Since $\operatorname{card}\overline{H}$ is invertible in $\mathbf{F}_p$, the representation $\chi\colon \overline{H} \to \mathrm{Aut}\langle w\rangle \cong \mathbf{F}_p^*$ is a direct summand of the representation $\overline{H} \le G \le GL_n(\mathbf{F}_p)$. It is then clear that the p-adic lifting of the representation $\overline{H} \le G \le GL_n(\hat{\mathbf{Z}}_p)$ has an invariant line generated by a vector $\hat{w} \in (\hat{\mathbf{Z}}_p)^n$ that reduces to $w \bmod p$.

Let $k\colon (B\mathbf{Z}/p^\infty)^n \to B\mathbf{Z}/p^\infty$ be the map induced by the group homomorphism $x \in (\mathbf{Z}/p^\infty)^n \mapsto x\hat{w} \in \mathbf{Z}/p^\infty$. Now, $\overline{H}$ acts on $(B\mathbf{Z}/p^\infty)^n$ while $\mathbf{F}_p^*$ acts on $B\mathbf{Z}/p^\infty$ and since for any $h \in \overline{H}$, $h\hat{w} = \hat{w}\cdot\chi(h)$, k becomes an equivariant map, so it induces

$$\tilde{k}\colon (B\mathbf{Z}/p^\infty)^n \times_{\overline{H}} E\overline{H} \to B\mathbf{Z}/p^\infty \times_{\mathbf{Z}/p-1} E\mathbf{Z}/p-1.$$

If t is the generator of the mod p cohomology of $B\mathbf{Z}/p^\infty$, then $k^*(t) = w$ and the map induced by $\tilde{k}$ in mod p cohomology is determined by $\tilde{k}^*(t^{p-1}) = w^{p-1} \in P(V^*)^{\overline{H}} \cong H^*((B\mathbf{Z}/p^\infty)^n \times_{\overline{H}} E\overline{H}; \mathbf{F}_p)$.

Now we pick the $\hat{S}_p^3$-fibration over $B\mathbf{Z}/p^\infty \times_{\mathbf{Z}/p-1} E\mathbf{Z}/p-1$ of Proposition 2. Then, the fibration over $(B\mathbf{Z}/p^\infty)^n \times_{\overline{H}} E\overline{H}$ obtained as a pull-back of that one has characteristic classes $q = 1 + w^{p-1}$.

The fibration we wish over $(B\mathbf{Z}/p^\infty)^n \times_G EG$ is obtained as a transfer of the previous one.

Let $\overline{X}$ denote $(B\mathbf{Z}/p^\infty)^n \times EG$ and

$$\Phi\colon \overline{X}/G \to (\overline{X}/\overline{H})^m \times_{\Sigma_m} E\Sigma_m$$

the pretransfer defined by Kahn–Priddy [4]: a representation $\rho\colon G \to \Sigma_m$ is defined by $g\tau_i = \tau_{\rho(g)(i)}h_i$, $h_i \in \overline{H}$, so that

$$\overline{X} \times EG \to (\overline{X}/\overline{H})^m \times E\Sigma_m\,,$$
$$(x\,, k) \mapsto (\overline{x\tau_1}\,, \ldots\,, \overline{x\tau_m}\,; \rho_*(k))$$

is equivariant. Φ is the induced map between the quotient spaces.

The structural map of the transfer is defined as follows. First of all we decompose $\hat{S}_p^{4m-1}$ as

$$\hat{S}_p^{4m-1} \simeq (\overbrace{\hat{S}_p^3 * \cdots * \hat{S}_p^3}^{m})_p^\wedge \simeq (\overbrace{\hat{S}_p^3 \times \cdots \times \hat{S}_p^3}^{m} \times \Delta_{m-1}/\sim)_p^\wedge$$

where $\Delta_{m-1} = \{(t_1, \dots, t_m) \mid 0 \leq t_i, \ \sum_{i=1}^m t_i = 1\}$ is the standard $(m-1)$-simplex, and the identifications are

$$(x_1, \dots, x_i, \dots, x_m; t_1, \dots, \overset{i}{\hat{0}}, \dots, t_m)$$
$$\sim (x_1, \dots, x_i', \dots, x_m; t_1, \dots, \overset{i}{\hat{0}}, \dots, t_m).$$

This allows us to describe a morphism of topological monoids:

$$\mathrm{Aut}(\hat{S}_p^3) \wr \Sigma_m \xrightarrow{\theta} \mathrm{Aut}(\hat{S}_p^{4m-1}),$$

where $\mathrm{Aut}(X)$ means the topological monoid of self-homotopy equivalences of X and $\wr$ stands for wreath product. $\theta(f_1, \dots, f_m; 1)$ is induced by

$$(x_1, \dots, x_m; t_1, \dots, t_m) \mapsto (f_1(x_1), \dots, f_m(x_m); t_1, \dots, t_m)$$

and $\theta(1, \dots, 1; \sigma)$ is induced by

$$(x_1, \dots, x_m; t_1, \dots, t_m) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(m)}; t_{\sigma^{-1}(1)}, \dots, t_{\sigma^{-1}(m)}).$$

So, we may define

$$s: (B\,\mathrm{Aut}(\hat{S}_p^3))^m \times_{\Sigma_m} E\Sigma_m \simeq B(\mathrm{Aut}(\hat{S}_p^3) \wr \Sigma_m) \xrightarrow{B\theta} B\,\mathrm{Aut}(\hat{S}_p^{4m-1}).$$

Finally, if the mod p spherical fibration that we have over $(B\mathbf{Z}/p^\infty)^n \times_{\overline{H}} E\overline{H}$ is classified by a map

$$h: (B\mathbf{Z}/p^\infty)^n \times_{\overline{H}} E\overline{H} \simeq \overline{X}/\overline{H} \to B\,\mathrm{Aut}(\hat{S}_p^3),$$

we obtain its transfer fibration $\zeta \downarrow \overline{X}/G$ as the one classified by

$$f_\zeta: X = \overline{X}/G \xrightarrow{\Phi} (\overline{X}/\overline{H})^m \times_{\Sigma_m} E\Sigma_m \xrightarrow{h^m \times 1} (B\,\mathrm{Aut}(\hat{S}_p^3))^m \times_{\Sigma_m} E\Sigma_m$$
$$\xrightarrow{s} B\,\mathrm{Aut}(\hat{S}_p^{4m-1}).$$

But we want to know something about orientability. Since the fibration over $\overline{X}/\overline{H}$ is orientable, h lifts to $(BGL_4)_p^\wedge$, the universal covering of $B\,\mathrm{Aut}(\hat{S}_p^3)$ [6]. Now the composition

$$(BGL_4)_p^\wedge \times_{\Sigma_m} E\Sigma_m \to (B\,\mathrm{Aut}(\hat{S}_p^3))^m \times_{\Sigma_m} E\Sigma_m \to B\,\mathrm{Aut}(\hat{S}_p^{4m-1})$$

maps the fundamental group of the space on the left, Σ_m, trivially to the fundamental group of $B\,\mathrm{Aut}(\hat{S}_p^{4m-1})$. In fact, we may as well think of $\theta(1, \dots, 1; \sigma)$ as the map induced by the orthogonal transformation $(v_1, \dots, v_m) \in (\mathbf{R}^4)^m \mapsto (v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(m)}) \in (\mathbf{R}^4)^m$, and this is an orientation preserving map. As a consequence we have a lifting:

$$\begin{array}{ccc} (BGL_4)_p^\wedge \times_{\Sigma_m} E\Sigma_m & \longrightarrow & (BGL_{4m})_p^\wedge \\ \downarrow & & \downarrow \\ (B\,\mathrm{Aut}(\hat{S}_p^3))^m \times_{\Sigma_m} E\Sigma_m & \longrightarrow & B\,\mathrm{Aut}(\hat{S}_p^{4m-1}) \end{array}$$

and then $\zeta \downarrow \overline{X}/G$ is actually orientable.

By construction of Φ the characteristic classes of ζ are calculated as

$$q(\zeta) = \prod_{i=1}^m (1 + \tau_i w^{p-1}) = \prod_{i=1}^m (1 + w_i^{p-1}). \quad \square$$

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