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ON AN EXTENSION OF MINTY'S THEOREM

MAKOTO SHIMIZU

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ABSTRACT. The notion of monotone operator in a Hilbert space is extended, and a considerably broader class of possibly multivalued operators is introduced. It is shown that well-known Minty's theorems on maximal monotone operators in Hilbert spaces can be extended to the cases of operators belonging to the class. Typical examples of partial differential operators are given to illustrate that the results can be applied to nonlinear equations, which involve nonmonotone differential operators.

In 1962 Minty [4] gave the following two important results:

(I) A monotone operator A in a Hilbert space H is maximal if and only if $R(I+\lambda A)=H$ for some $\lambda>0$.

(II) If a monotone operator A in a Hilbert space H is defined on all of H and is continuous, then A is maximal monotone.

These results have been applied to show the existence of solutions of a variety of partial differential equations, especially nonlinear elliptic equations. Since then these results have been generalized in various directions to Banach spaces (see for instance Barbu [1]). However, to the best of the author's knowledge, it seems that Minty's theorems have not been extended to the case of nonmonotone operators even though they are linear.

In this paper we extend the above results to a class of nonmonotone operators in Hilbert spaces. This class, denoted herein by $\mathscr{M}(\alpha)$, $0\le\alpha<1$, includes monotone operators as a special class. Our arguments are based on the method exploited in Kōmura [3] (cf. [2]).

In §1 we introduce classes of multivalued operators, $\mathscr{M}(\alpha)$, $\alpha\in[0,1)$, and state our main theorems. In §2 we give the proofs after preparing two lemmas. Section 3 contains several remarks on perturbations as well as an example of a linear partial differential operator that is elliptic but not strongly elliptic.

1. DEFINITIONS AND MAIN RESULTS

Let H be a complex Hilbert space with norm $|\cdot|$ and inner-product $\langle\cdot,\cdot\rangle$. By a multivalued operator in H we mean an operator that assigns to each

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$x \in H$ a (possible empty) subset of H. Such an operator is identified as its graph and it is viewed as a subset of $H \times H$. For $A \subset H \times H$, we define

$$Ax = \{y : [x, y] \in A\},$$
$$D(A) = \{x : Ax \neq \varnothing\}, \text{ and}$$
$$R(A) = \bigcup\{Ax : x \in D(A)\}.$$

If A, $B \subset H \times H$, and λ is complex, then one sets

$$A + B = \{[x, y + z] : y \in Ax \text{ and } z \in Bx\}, \text{ and}$$
$$\lambda A = \{[x, \lambda y] : y \in Ax\}, \quad A^{-1} = \{[y, x] : [x, y] \in A\}.$$

Then we introduce the following class of operators as an extension of monotone operators. Let $\alpha \in [0, 1)$. An operator A is said to be of the class $\mathscr{M}(\alpha)$ if and only if

$$(M; \alpha) \qquad \operatorname{Re}\langle x_1' - x_2', x_1 - x_2\rangle \geq -\alpha|x_1' - x_2'||x_1 - x_2| \quad \text{for every pair } [x_i, x_i'] \in A, \ i = 1, 2.$$

Note that if α, $\beta \in [0, 1)$ with $\alpha \leq \beta$ then $\mathscr{M}(\alpha)$ is included in $\mathscr{M}(\beta)$, in the set theoretical sense, and that A is monotone if and only if $A \in \mathscr{M}(0)$.

Let $A \in \mathscr{M}(\alpha)$. A is said to be maximal in the class $\mathscr{M}(\alpha)$ if it is not properly contained in any other operator in the class $\mathscr{M}(\alpha)$. This notion of maximality is a straightforward generalization of the maximality of monotone operators.

We now state our main results. The first result extends Minty's Theorem (I).

Theorem 1. *Let $A \in \mathscr{M}(\alpha)$. Then A is maximal in the class $\mathscr{M}(\alpha)$ if and only if $R(I + \lambda A) = H$ for some $\lambda > 0$.*

As an extension of Minty's second theorem (II) we obtain the following result:

Theorem 2. *Let $A \in \mathscr{M}(\alpha)$. If $D(A) = H$ and A is continuous along line segments (i.e., hemicontinuous), then A is maximal in the class $\mathscr{M}(\alpha)$.*

2. Proofs of theorems

In order to prove Theorems 1 and 2, we need two lemmas.

Lemma 1. *An operator A is of the class $\mathscr{M}(\alpha)$ if and only if for $\lambda > 0$ and for $[x_i, x_i'] \in A$, $i = 1, 2$, we have*

$$(2.1) \qquad |x_1 - x_2 - \lambda(x_1' - x_2')| \leq \left(\frac{1+\alpha}{1-\alpha}\right)^{1/2} |x_1 - x_2 + \lambda(x_1' - x_2')|.$$

Proof. First suppose that A is of the class $\mathscr{M}(\alpha)$. Then for any pair $[x_i, x_i'] \in A$, $i = 1, 2$, we have

$$|x_1 - x_2 \pm \lambda(x_1' - x_2')|^2 = |x_1 - x_2|^2 + |\lambda(x_1' - x_2')|^2 \pm 2\operatorname{Re}\langle \lambda(x_1' - x_2'), x_1 - x_2\rangle,$$

respectively. It follows from the assumption that

$$|x_1 - x_2 - \lambda(x_1' - x_2')|^2 - 2\alpha|\lambda(x_1' - x_2')||x_1 - x_2| \leq |x_1 - x_2|^2 + |\lambda(x_1' - x_2')|^2$$
$$\leq |x_1 - x_2 + \lambda(x_1' - x_2')|^2 + 2\alpha|\lambda(x_1' - x_2')||x_1 - x_2|.$$

Hence

$$(2.2)\quad |x_1 - x_2 - \lambda(x_1' - x_2')|^2 \le |x_1 - x_2 + \lambda(x_1' - x_2')|^2 + 4\alpha|\lambda(x_1' - x_2')||x_1 - x_2|.$$

Now, the application of the parallelogram law yields

$$\begin{aligned} 2|\lambda(x_1' - x_2')||x_1 - x_2| &\le |\lambda(x_1' - x_2')|^2 + |x_1 - x_2|^2 \\ &\le \tfrac{1}{2}\{|x_1 - x_2 - \lambda(x_1' - x_2')|^2 + |x_1 - x_2 + \lambda(x_1' - x_2')|^2\}. \end{aligned}$$

This estimate and (2.2) together imply

$$\begin{aligned} |x_1 - x_2 - \lambda(x_1' - x_2')|^2 &\le |x_1 - x_2 + \lambda(x_1' - x_2')|^2 \\ &\quad + \alpha\{|x_1 - x_2 - \lambda(x_1' - x_1')|^2 + |x_1 - x_2 + \lambda(x_1' - x_2')|^2\}. \end{aligned}$$

From this the desired inequality (2.1) follows.

Conversely, assume that (2.1) holds. Since condition $(M;\alpha)$ is clearly satisfied for $[x_i, x_i'] \in A$, $i = 1, 2$, provided $\operatorname{Re}\langle x_1' - x_2', x_1 - x_2\rangle \ge 0$, it is sufficient to show that $(M;\alpha)$ holds in the case where $\operatorname{Re}\langle x_1' = x_2', x_1 - x_2\rangle < 0$. It follows from (2.1) that for any $\lambda > 0$

$$\lambda^2(\alpha|x_1' - x_2'|)^2 + 2\lambda \operatorname{Re}\langle x_1' - x_2', x_1 - x_2\rangle + \alpha|x_1 - x_2|^2 \ge 0.$$

Therefore we have

$$|\operatorname{Re}\langle x_1' - x_2', x_1 - x_2\rangle|^2 - \alpha^2|x_1' - x_2'|^2|x_1 - x_2|^2 \le 0.$$

Since $\operatorname{Re}\langle x_1' - x_2', x_1 - x_2\rangle < 0$, this means that condition $(M;\alpha)$ holds. □

In a way similar to the proof of Lemma 1, we obtain the following result:

Corollary. *An operator A is of class $\mathscr{M}(\alpha)$ if and only if for any $\lambda > 0$ and any pair $[x_i, x_i'] \in A$, $i = 1, 2$, the inequality holds*

$$(2.3)\qquad |x_1 - x_2| \le (1 - \alpha^2)^{-1/2}|x_1 - x_2 + \lambda(x_1' - x_2')|.$$

Next, let A be of the class $\mathscr{M}(\alpha)$ and let $\lambda > 0$. We define an operator $S_{\lambda A}\colon R(I + \lambda A) \to H$ by

$$(2.4)\qquad S_{\lambda A}(x + \lambda x') = x - \lambda x'$$

for $[x, x'] \in A$. By Lemma 1, $S_{\lambda A}$ is a Lipschitz continuous operator with Lipschitz constant $L = ((1+\alpha)/(1-\alpha))^{1/2}$.

Lemma 2. *Let A, B be of the class $\mathscr{M}(\alpha)$ and let $S_{\lambda A}$, $S_{\lambda B}$ be the corresponding operators defined respectively by (2.4). Then $A \subset B$ if and only if $S_{\lambda A} \subset S_{\lambda B}$.*

Proof. Assume $A \subset B$. Let $y \in D(S_{\lambda A})$, then there exists $[x, x'] \in A$ with $y = x + \lambda x'$. By assumption, $y \in D(S_{\lambda B})$ and $S_{\lambda B}y = x - \lambda x' = S_{\lambda A}y$. This means that $S_{\lambda A} \subset S_{\lambda B}$. Conversely, suppose that $S_{\lambda A} \subset S_{\lambda B}$. Let $[x, x'] \in A$. Then $y = x + \lambda x' \in D(S_{\lambda A}) \subset D(S_{\lambda B})$. Hence, there exists $[z, z'] \in B$ with $y = z + \lambda z'$. From this it follows that $z + \lambda z' = x + \lambda x'$ and $z - \lambda z' = S_{\lambda B}y = S_{\lambda A} = x - \lambda x'$, and these identities together imply $[x, x'] = [z, z'] \in B$. □

We infer from Lemmas 1 and 2 that an operator A can be extended to an operator in the class $\mathscr{M}(\alpha)$ if and only if the operator $S_{\lambda A}$ can be extended to a Lipschitz continuous operator with Lipschitz constant $L = ((1+\alpha)/(1-\alpha))^{1/2}$.

It should be noted that the operator A has a maximal extension $\tilde{A}$ in the class $\mathscr{M}(\alpha)$ by the routine method used the Zorn's lemma. Then the maximality

of $S_{\lambda\widetilde{A}}$, which follows from Lemma 2, implies that $S_{\lambda\widetilde{A}} = \widetilde{S}_{\lambda A}$ where $\widetilde{S}_{\lambda A}$ is the maximal extension of $S_{\lambda A}$ with $D(\widetilde{S}_{\lambda A}) = H$ in the sense of Theorem LE as stated below.

Proof of Theorem 1. Assume that A is maximal in the class $\mathscr{M}(\alpha)$. Then for any $\lambda > 0$, $S_{\lambda A}$ is maximal with $D(S_{\lambda A}) = H$. This means that $R(I+\lambda A) = H$ for any $\lambda > 0$. Conversely, suppose that $R(I + A) = H$ for some $\lambda > 0$. Then $S_{\lambda A}$ is maximal and hence A is maximal. □

Remarks. 1. From the proof of Theorem 1 we see that $R(I + \lambda A) = H$ for all $\lambda > 0$ if and only if $R(I + \lambda A) = H$ for some $\lambda > 0$.

2. We used the following well-known fact in the proof of Theorem 1 on the Lipschitz extension.

Theorem LE. *Let H be a Hilbert space and let T be an operator satisfying $|Tx - Ty| \le L|x - y|$ for x, $y \in D(T)$ where L is a constant. Then there exists an operator $\widetilde{T}: H \to H$ such that $\widetilde{T} \supset T$ and $|\widetilde{T}x - \widetilde{T}y| \le L|x - y|$ for x, $y \in H$.*

For the proof we refer to [5, p. 21].

Proof of Theorem 2. Assume that the operator A is not maximal in the class $\mathscr{M}(\alpha)$. Since $D(A) = H$, A has an extension $\widetilde{A}$ in the class $\mathscr{M}(\alpha)$ with $D(\widetilde{A}) = H$, and there exist x, y, $\tilde{y} \in H$ such that $y = Ax$, $\tilde{y} \in \widetilde{A}x$, and $y \ne \tilde{y}$. Put $x_t = x + t(\tilde{y} - y)$, for $t > 0$, and let $x_t \to x$ and $Ax_t \to y$ as $t \downarrow 0$. Then

$$(1/t)\{\mathrm{Re}\langle Ax_t - \tilde{y}, x_t - x\rangle + \alpha|Ax_t - \tilde{y}||x_t - x|\} \ge 0,$$

and the left-hand side converges to $(\alpha - 1)|y - \tilde{y}|^2$ as $t \downarrow 0$. Therefore $y = \tilde{y}$. This is a contradiction, and A must be maximal in the class $\mathscr{M}(\alpha)$. □

3. *Remarks.*

(I) As a consequences of Theorem 1, we obtain the following perturbation theorem for operators in $\mathscr{M}(\alpha)$.

Theorem 3. *Let A be a maximal in the class $\mathscr{M}(\alpha)$ and B a Lipschitz continuous operator with domain $D(B) = H$ and Lipschitz constant β.*

(i) *If $A + B \in \mathscr{M}(\gamma)$ for some $\gamma \in [0, 1)$, then $A + B$ is maximal in the class $\mathscr{M}(\gamma)$.*

(ii) *If B is monotone and there exists $\delta \in (0, 1/\alpha)$ such that*

$$|x_1' - x_2'| \le \delta|x_1' - x_2' + Bx_1 - Bx_2| \quad \textit{for } [x_i, x_i'] \in A, \ i = 1, 2,$$

then $A + B$ is of the class $\mathscr{M}(\alpha\delta)$.

Proof. (i) In view of Theorem 1, it is sufficient to show that the equation $u + \lambda(A + B)u \ni f$ has a solution for $f \in H$ and some $\lambda > 0$. To this end, let $f \in H$ and define an operator $T: H \to H$ by $Tg = (I + \lambda A)^{-1}(f - \lambda Bg)$ for $g \in H$. By Corollary to Lemma 1, T is a strict contraction for λ satisfying $0 < \lambda\beta/(1 - \alpha^2)^{1/2} < 1$. Therefore T has a fixed point in $D(A)$ and this fixed point gives a solution of the above equation. Assertion (ii) follows immediately from the definition of our class of operators $\mathscr{M}(\alpha)$ in H. □

(II) Finally, we give two examples of differential operators that belong to the class of $\mathscr{M}(\alpha)$. First we consider a linear partial differential operator with constant coefficients

$$A = \sum_{|\beta|=2m} a_\beta \cdot (\partial^\beta / \partial x^\beta), \qquad a_\beta \in C,$$

where β is a multi-index and m is a positive integer. Let $H = L^2(R^n)$ with norm $\| \ \|$ and $D(A) = H^{2m}(R^n)$. The operator A satisfies condition $(M; \alpha)$ for some $\alpha \in [0, 1)$, if the next condition holds.

$$\text{(3.1)} \qquad \operatorname{Re}\left((-1)^m \sum_{|\beta|=2m} a_\beta \xi^\beta\right) \geq -\alpha \left| \sum_{|\beta|=2m} a_\beta \xi^\beta \right| \quad \text{for } \xi \in R^n.$$

Indeed, if we assume (3.1), then we get by the Parseval's identity and the Cauchy-Schwarz inequality

$$\begin{aligned}
\operatorname{Re}\langle Au, u\rangle &= \operatorname{Re} \int_{R^n} \sum_{|\beta|=2m} a_\beta \cdot (i\xi)^\beta \cdot \hat{u}(\xi)\overline{\hat{u}(\xi)}\, d\xi \\
&= \operatorname{Re} \int_{R^n} (-1)^m \sum_{|\beta|=2m} a_\beta \cdot \xi^\beta \cdot |\hat{u}(\xi)|^2\, d\xi \\
&\geq -\alpha \int_{R^n} \left| \sum_{|\beta|=2m} a_\beta \cdot \xi^\beta \right| |\hat{u}(\xi)|^2\, d\xi \\
&\geq -\alpha \|\widehat{Au}\| \, \|\hat{u}\| = -\alpha \|Au\| \, \|u\|,
\end{aligned}$$

where $\hat{u}$ is the Fourier transform of u.

In particular, let A_0 be the differential operator $(\partial/\partial x)^2 - (\partial/\partial y)^2 + i(\partial/\partial x)^2$ with domain $D(A_0) = H^2(R^2)$. Note that the operator A_0 is elliptic but not strongly elliptic. A simple calculation yields

$$-\xi_1^2 + \xi_2^2 \geq -2^{-1/2}\{(\xi_1^2 - \xi_2^2)^2 + \xi_1^2\}^{1/2}$$

for $(\xi_1, \xi_2) \in R^2$, and this shows that the condition (3.1) holds with $\alpha = 2^{-1/2}$.

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DEPARTMENT OF MATHEMATICS, SCHOOL OF EDUCATION, WASEDA UNIVERSITY, 1-6-1, NISI-WASEDA, SHINJUKU-KU, TOKYO 169, JAPAN

Current address: 4-2-204 Niiza 3-choume, Niiza Saitama 352, Japan