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BEST APPROXIMATION BY SUBHARMONIC FUNCTIONS

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ABSTRACT. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. We prove existence of best subharmonic approximations in $L_\infty(\Omega)$ and, for functions continuous in $\overline{\Omega}$, we characterize best continuous subharmonic approximations.

0. INTRODUCTION

In this paper we study the problem of best approximation in the norm of L_∞ on a bounded domain $\Omega \subset \mathbb{R}^d$ by subharmonic functions. We show that for arbitrary elements of $L_\infty(\Omega)$, such approximations exist, and for a continuous function, we characterize best continuous subharmonic approximations (i.e., best approximations from the set of functions subharmonic in Ω and continuous in $\overline{\Omega}$).

The choice of $L_\infty(\Omega)$ poses a challenge due to its lack of uniform convexity and reflexivity, yet it is a natural setting for such approximation problems [2]. An additional complicating factor is that the set S of subharmonic functions defined on a domain of $\mathbb{R}^d$ is not a subspace; rather, it is a convex cone with vertex $S \cap (-S)$ comprising the harmonic functions.

Subharmonic functions, which play a central role in both classical and modern potential theory, are perhaps most familiar to students of partial differential equations from the Perron, or PWB, method of constructing a solution to the Dirichlet problem [6, 7].

In problems for which we know a priori that the exact solution must be subharmonic, it makes sense to seek numerical approximations that have this property as well. By characterizing best approximations to a given function by subharmonic functions, we provide a theoretical underpinning on which to base both the search for efficient numerical methods and the evaluation of their effectiveness.

Subharmonic functions are a natural generalization to several variables of univariate convex functions and share many of their convenient attributes. In contrast to multivariate convex functions, they are characterized by an integral

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representation involving a positive Borel measure. In addition, they satisfy the familiar Mean Value Inequality: the value of a subharmonic function at the center of a ball is at most equal to its integral mean over the surface of the ball. Moreover, and most important in subharmonic approximation problems, they remain subharmonic under harmonic lifting, the "pushing up" of values that occurs when a subharmonic function is replaced on an open subset by the harmonic function that agrees with it on the boundary of that set.

The study of best uniform approximation to continuous functions by harmonic functions was initiated by Burchard [1], who proved an existence result and a characterization theorem for the unit ball. Similar results for arbitrary Jordan domains were obtained independently by the authors of [8]. These papers reveal how to generalize the one-dimensional notion of an alternant, a set of points on which the maximal deviation is achieved with alternating sign. This insight, and the experience gained in studying approximation by convex functions [12], eventually led to the results presented in this paper.

1. Preliminaries

There is a wealth of information in the literature on subharmonic functions and their properties. In particular, the reader is referred to the books [4, 6, 7, 9].

Let Ω be a domain (a connected, open set) in $\mathbb{R}^d$. A function $g\colon \Omega \mapsto [-\infty, \infty)$ is *subharmonic* if it is upper semicontinuous, not identically $-\infty$, and, for all balls $B = B(x_0, r)$ such that $\overline{B} \subset \Omega$, satisfies the Mean Value Inequality [7]:

$$g(x_0) \leq \frac{1}{|\partial B|} \int_{\partial B} g(x)\, d\sigma(x),$$

where $d\sigma$ denotes the $(n-1)$-dimensional area element on ∂B and $|\partial B|$ is the surface measure of ∂B. If g is in $C^2(\Omega)$ then g is subharmonic if and only if

$$\Delta g = \frac{\partial^2 g}{\partial x_1^2} + \cdots + \frac{\partial^2 g}{\partial x_n^2} \geq 0.$$

We note that subharmonic functions are locally integrable. In the sense of distributions [10], a subharmonic function g is represented by its Laplacian, which is a nonnegative distribution, i.e., a Radon measure:

$$\langle \Delta g, \phi \rangle := \int g \Delta \phi,$$

for all test functions $\phi \in C_0^\infty(\Omega)$. Conversely, if g is a distribution such that $\Delta g \geq 0$, then g is defined by a subharmonic function.

A function is *superharmonic* if its negative is subharmonic; a function that is both subharmonic and superharmonic is *harmonic*. Harmonic functions satisfy Laplace's equation, $\Delta g = 0$. We observe that a function of one variable is subharmonic if and only if it is convex and that univariate harmonic functions are simply linear polynomials.

We recall that the Dirichlet problem for an open set Ω consists of finding a harmonic function u such that $\lim_{x \to \xi} u(x) = f(\xi)$, $x \in \Omega$, $\xi \in \partial\Omega$, for a given boundary function f. A point $\xi \in \partial\Omega$ is called "regular" or "regular for the Dirichlet problem" if this condition is satisfied for all bounded, measurable

functions f that are continuous at ξ. If every point of the boundary is regular then Ω is called regular. It is well known [9] that a point $\xi \in \partial\Omega$ is regular precisely when there is a *barrier* at ξ, superharmonic function w defined in a neighborhood W of ξ such that $w > 0$ in $W \cap \Omega$ and $\lim_{x\to\xi} w(x) = 0$, $x \in W \cap \Omega$. The *harmonic lifting* $\hat{g}$ of a subharmonic function g in a regular domain W with $\overline{W} \subset \Omega$ coincides with the solution of the Dirichlet problem on W with g as boundary function and equals g outside of W. By the Maximum Principle, $\hat{g} > g$ in W unless g is harmonic in W.

Let S denote the set of subharmonic functions in $L_\infty(\Omega)$. An element u in S is a *best subharmonic approximation* to $f \in L_\infty(\Omega)$ if

$$\|f - u\|_\infty = \operatorname*{ess\,sup}_{\Omega} |f - u| = \inf_{v\in S} \|f - v\|_\infty .$$

The difficulty in finding a best subharmonic approximation to a given function f is that $\rho = \inf_{v\in S} \|f - v\|_\infty$ is, in general, not known in advance. What is required is a simple test, such as that given in Theorem 2, that determines if a given $u \in S$ is optimal and at the same time yields the optimal value of ρ. The problem at hand is akin to that of finding best convex approximations in the one-dimensional case, and our approach is based on the proofs given in [12].

2. Main results

In this section we prove the existence of best subharmonic approximations in L_∞ and characterize best continuous subharmonic approximations. It is worth keeping in mind that a best continuous subharmonic approximation may not be a best subharmonic approximation; however, in [13] it is shown that a continuous function on a compact regular domain has a continuous best subharmonic approximation.

We note that the proof of sufficiency is a straightforward application of the Maximum Principle for subharmonic functions, valid in arbitrary bounded domains, and that under the conditions of the theorem, all best approximations agree on a certain open set E_-. Thus E_- generalizes the idea of a "fundamental" or "core" subinterval, familiar from univariate uniform approximation [5].

Theorem 1. *Let Ω be an open subset of $\mathbb{R}^d$. Then every function in $L_\infty(\Omega)$ has a best subharmonic approximation.*

Proof. This follows from a standard functional analytic argument, which we sketch here for the sake of completeness.

Let f be an element of $L_\infty(\Omega)$, and let S denote the subset of subharmonic functions in $L_\infty(\Omega)$. It suffices to show that S is weak* closed in $L_\infty(\Omega)$ since this ensures that S is *proximal*, i.e., closest points to external points exist [3].

Let $\{g_n\}$ be a sequence in S such that $g_n \to g$ weak* in $L_\infty(\Omega)$, i.e.,

$$\int g_n\phi \to \int g\phi, \qquad \text{for all } \phi \in L_1(\Omega).$$

In particular, if ϕ is a nonnegative element of $C_0^\infty(\Omega)$, then

$$0 \le \langle \Delta g_n , \phi\rangle = \int g_n\Delta\phi \to \int g\Delta\phi = \langle \Delta g , \phi\rangle ,$$

so that Δg is a nonnegative distribution. Thus g is a subharmonic function and S is weak* closed, completing the proof of existence. □

We mention here another approach to proving the existence of best subharmonic approximations, utilizing the method introduced in [11]. With this approach, one constructs the greatest subharmonic minorant [4] $\tilde{g}$ to f on Ω, which is the upper semicontinuous smoothing of

$$\sup\{\hat{g}\colon \hat{g} \text{ is subharmonic and } \hat{g} \le f\}.$$

Letting $\rho := \|f - \tilde{g}\|_\infty$, one then shows that $g := \tilde{g} + \frac{1}{2}\rho$ is a best subharmonic approximation (it is in fact the "upper envelope" of all such approximations). Our characterization of best subharmonic approximations thus also sheds light on the nature of greatest subharmonic minorants. For further investigations along these lines, see [13].

Theorem 2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let f and g be continuous in $\overline{\Omega}$ with g subharmonic in Ω (we exclude the trivial case in which f is subharmonic in Ω). Define*

$$K_+ := \{x \in \overline{\Omega}\colon f(x) - g(x) = +\|f - g\|_\infty\},$$
$$K_- := \{x \in \overline{\Omega}\colon f(x) - g(x) = -\|f - g\|_\infty\}.$$

If there exists a domain $E_- \subset \Omega$ with $\partial E_- \subset K_-$ such that g is harmonic in E_- and $K_+ \cap E_- \neq \varnothing$, then g is a best continuous subharmonic approximation to f and all best continuous subharmonic approximations to f agree on E_-. Conversely, if Ω is regular for the Dirichlet problem then there exists a domain $E_- \subset \Omega$ with $\partial E_- \subset K_-$ such that g is harmonic in E_- and $K_+ \cap E_- \neq \varnothing$.

Proof. Sufficiency: Suppose that g_1 is any other function that is subharmonic in Ω and continuous in $\overline{\Omega}$ satisfying

$$\|f - g_1\|_\infty \le \|f - g\|_\infty.$$

Then, necessarily,

$$(f - g_1) \le (f - g) \quad \text{on } K_+,$$
$$(f - g_1) \ge (f - g) \quad \text{on } K_-.$$

Thus, for $x_0 \in K_+ \cap E_-$, we have $(g_1 - g)(x_0) \ge 0$ and $(g_1 - g)(x) \le 0$ for all $x \in \partial E_-$. Since $g_1 - g$ is subharmonic in E_- and continuous in $E_- \cup \partial E_-$, by the Maximum Principle we must have $g_1 - g \equiv 0$ in E_-. This shows that g is a best approximation to f and that all best approximations agree on E_-.

Necessity: In order to prove necessity we first need the following results.

Lemma 1. *If Ω_1, Ω_2 are regular domains is $\mathbb{R}^d$, so too is $\Omega_1 \cap \Omega_2$.*

Proof. We need to show that we can construct a barrier at each $x \in \partial(\Omega_1 \cap \Omega_2)$. Clearly,

$$\begin{aligned}\partial(\Omega_1 \cap \Omega_2) &\subseteq (\Omega_1 \cap \partial\Omega_2) \cup (\Omega_2 \cap \partial\Omega_1) \cup (\partial\Omega_1 \cap \partial\Omega_2)\\ &=: A \cup B \cup C.\end{aligned}$$

Cases A and B are trivial (recall, we are assuming that we can construct barriers in Ω_1 and Ω_2), simply let the barriers be those for Ω_2 and Ω_1, respectively. In case C, let it be the minimum of the two barriers. □

Lemma 2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, regular domain. Let f be continuous in $\overline{\Omega}$. If g is a best continuous subharmonic approximation to f then $K_+ \cap \Omega \neq \varnothing$.*

Proof. We may assume that $\|f - g\|_\infty := \rho > 0$ so that

$$K_\pm = \{z \in \overline{\Omega}\colon f(z) - g(z) = \pm\rho\}.$$

Clearly K_+ is nonempty; otherwise, the norm of $f - g$ may be reduced by adding a constant to g. We need to show that not all of K_+ lies on $\partial\Omega$.

Suppose, to the contrary, that $K_+ \subset \partial\Omega$. Cover K_+ with open balls $B_1, \dots, B_N$ of radius ε small enough so that $f - g > \rho/2$ in $B := \bigcup B_i$. Let $A_1, \dots, A_J$ be a finite number of balls of radius $\delta < \varepsilon$ whose union A satisfies $K_+ \subset A \subset \overline{A} \subset B$. By [9, p. 175] there is a regular open set C such that $\overline{A} \subset C \subset B$. Let $C_1, \dots, C_M$ be the (regular) connected components of C; by construction, there are only finitely many of them.

By Urysohn's Lemma, there exist continuous functions $\psi_i\colon C_i \mapsto [-1, 1]$ for each i such that $\psi_i \equiv 1$ on $K_+ \cap C_i$ and $\psi_i \equiv -1$ on ∂C_i. By Lemma 1 we can solve the Dirichlet problem on each $C_i \cap \Omega$, where our boundary function is the restriction of ψ_i to $\partial(C_i \cap \Omega)$. Let us call the solutions we obtain u_i.

For $x \in \Omega$, set

$$\phi = \begin{cases} u_i & \text{in } C_i, \\ -1 & \text{otherwise}. \end{cases}$$

Since ϕ satisfies the Mean Value Inequality, it is subharmonic on Ω (see [6, p. 24]) and continuous on $\overline{\Omega}$. Note that $\phi \equiv -1$ on K_- and $\phi \equiv +1$ on K_+. By a standard argument (see, e.g., [8, p. 321]), for small enough $\varepsilon > 0$, $h := g + \varepsilon\phi$ will be a better approximation to f. Since h is continuous and subharmonic, this is a contradiction and Lemma 2 is proved □

Let us call any point $z_0 \in K_+ \cap \Omega$ a *max point*. An argument identical to the one given in [12] for best convex approximations now shows that there is a max point z_* common to *all* best continuous subharmonic approximations to f (an alternative approach to constructing z_* is given in [13, Corollary 6]).

We can now complete the proof of necessity. Let z_* be as in the preceding paragraph, and let g be a best continuous subharmonic approximation to f. We will show that there is a neighborhood V of z_* in Ω such that

(i) g is harmonic in V;
(ii) $f - g \equiv -\|f - g\|_\infty$ on ∂V.

Clearly g is harmonic in some neighborhood of z_*; for if not, we could replace g by its harmonic lifting $\hat{g}$ in a small ball around z_*. Since then $\hat{g}(z_*) > g(z_*)$, $\hat{g}$ would be a best continuous subharmonic approximation to f for which z_* would not be a max point.

Let us now set $\rho \equiv \|f - g\|_\infty$ as before. Let V be the largest connected open set containing z_* such that g is harmonic and $f - g > -\rho$ in all of V. We claim that $f(x) = g(x) = -\rho$ for all $x \in \partial V$. We shall prove this by contradiction, i.e., suppose $f(x) - g(x) > -\rho$ for some $z \in \partial V$. There are three cases to consider.

Case I. $x \in \Omega$. Center a small ball B_1 at x such that $\overline{B}_1 \subset \Omega$ and $f - g > -\rho$ in $\overline{B}_1$. Center another ball, B_2, at z_* such that $\overline{B}_2 \subset V$. Now connect B_1 and B_2 by a chain of balls $B_3, \dots, B_N$ satisfying $\overline{B}_i \subset V$ for

each $i = 3, \dots, N$ and so that the domain $D \equiv \bigcup_1^N B_i$ is regular. This is possible by [9, p. 175]. Notice that $\inf_D(f - g) > -\rho$.

Let us now solve the Dirichlet problem on D with the boundary data being the restriction of g. Call the solution u, and let $\hat{g}$ be the function that equals u on D and g elsewhere. If we choose the balls small enough, then as above, $\hat{g}$ will be a best continuous subharmonic approximation to f (we just have to guarantee that $f - \hat{g} \geq -\rho$). Moreover, since g cannot be harmonic in all of B_1 (by the maximality of V), we must have $\hat{g} > g$ in all of D. But then z_* is not a max point for $\hat{g}$, which is a contradiction. Therefore $f(x) - g(x) = -\rho$.

Case II. $x \in \partial\Omega$, but there is a ball B_1 centered at x such that $B_1 \cap \Omega \subset V$. Let B_1 be so small that $f - g > -\rho$ in $\overline{B}_1$. Use Urysohn's Lemma to get a function $\phi\colon \mathbb{R}^d \mapsto [0, 1]$ such that $\phi(x) = 1$ and $\phi \equiv 0$ outside B_1. Now let D be as in Case I and solve the Dirichlet problem with null boundary data on $\partial D \cap \Omega$ and ϕ on $\partial D \cap \partial\Omega$. Call the solution ψ. For small, positive ε set

$$\hat{g}_\varepsilon = \begin{cases} g + \varepsilon \cdot \psi & \text{in } D, \\ g & \text{otherwise}, \end{cases}$$

If ε and B_1 are small enough, $\hat{g}_\varepsilon$ will still be a best continuous subharmonic approximation to f. But by the Maximum Principle, $\hat{g}_\varepsilon > g$ in D, and so z_* is not a max point for $\hat{g}_\varepsilon$.

Case III. $x \in \partial\Omega$, and every ball B_1 containing x satisfies $B_1 \cap \Omega \not\subset V$. This is handled much like Case I. Let B_1 be a sufficiently small ball around x and construct D as before. According to Lemma 1, $\widetilde{D} := D \cap \Omega$ is regular. Now solve the Dirichlet problem on $\widetilde{D}$ with boundary data g. The resulting harmonic lifting of g in $\widetilde{D}$ will be a best continuous subharmonic approximation that does not have z_* as a max point.

This completes the proof of Theorem 2. □

As a first consequence of Theorem 2, we note that if, under the conditions on Ω stated in [8] or [1], a continuous function f has a best harmonic approximation g that is continuous in $\overline{\Omega}$ then either f or $-f$ (with $-g$) satisfies the conditions of Theorem 2 and, therefore, either f or $-f$ has a best continuous subharmonic approximation that is harmonic.

We also have the following result.

Corollary 1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. If f is continuous in $\overline{\Omega}$ and superharmonic in Ω and g is a best continuous subharmonic approximation to f, then g is harmonic in Ω and it is unique.*

Proof. Under these conditions $f - g$ is superharmonic, hence by the Maximum Principle its minimum is attained only on the boundary of Ω. It therefore follows from Theorem 2 that E_- is all of Ω, so that g is harmonic in Ω and is unique there. □

This result is also consistent with the fact that the greatest subharmonic minorant of a superharmonic function is harmonic. In the special case were f is superharmonic in Ω and continuous in $\overline{\Omega}$ and Ω is a bounded, regular domain, the PWB solution $\tilde{g}$ of the Dirichlet problem in Ω with f as boundary function has a continuous extension to $\overline{\Omega}$, and it is also the greatest subharmonic minorant of f in Ω. Thus, as demonstrated in [11], $g := \tilde{g} + \frac{1}{2}\|f - g\|_\infty$ is a best subharmonic approximation and is continuous in $\overline{\Omega}$.

If f is an arbitrary continuous function, the question of existence of continuous best approximations naturally arises. As was shown in [8], not every continuous function has a continuous best harmonic approximation, and it was expected that this would hold in the subharmonic case as well. However, it is possible to give conditions on f and Ω that guarantee the continuity or smoothness of best subharmonic approximations. This subject is treated in [13].

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