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NEW ESTIMATES FOR THE COLLAPSE OF THE MILNOR-MOORE SPECTRAL SEQUENCE OVER A FIELD

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ABSTRACT. We use the free tensor models of Halperin and Lemaire to give a new and transparent proof of a theorem of Ginsburg's on the collapse of the Milnor-Moore spectral under the assumption of finite L.-S. category. The method, valid over any field, provides better bounds for the collapse and these bounds are effectively computable L.-S. type invariants.

1. INTRODUCTION

A well-known result due originally to Ginsburg [Gi] states that if the Lusternik-Schnirelmann category $\operatorname{cat}(S)$ of a space S is m then the Milnor-Moore spectral sequence (hereafter M-M s.s) for S converging to $H(S)$ from $\operatorname{Ext}_{H*(\Omega S)}(\mathscr{k}, k)$ collapses at the $(m+1)$st stage. Ginsburg's was an intricate argument and later, Ganea [G] used his new characterization of L-S category to present another proof that simplified somewhat the difficult step in Ginsburg's proof, namely, the effect of finite category on a geometric analogue of the bar construction due to G. W. Whitehead.

Halperin and Lemaire have recently defined *free tensor models* [HL] and used them to define new L-S type homotopy invariants $\operatorname{Mcat}(S; k)$ and $\operatorname{Acat}(S; k)$ over any field k. These are much easier to compute than the classical $\operatorname{cat}(S)$ and provide lower bounds for it when S is simply connected. They can, for example, be computed fairly readily from an Adams-Hilton model for $C*(\Omega S)$, while $\operatorname{cat}(S)$ is usually difficult to calculate.

We use these free tensor models to prove the

Theorem. *Suppose S has the homotopy type of a simply connected CW complex of finite k-type. If $m = \operatorname{Mcat}(S; k)$, then the Milnor-Moore spectral sequence for S collapses at the $(m+1)$st stage.*

Note that this result is sharper than the classical theorem in the case

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$S = \mathrm{Sp}(2)$, where $\mathrm{Acat}(S, k) = 2$ and $\mathrm{cat}(S) = 3$. See [HL, S] for details.

2. Notation and definition of $\mathrm{Mcat}(S; k)$

We shall work with the cohomology M-M s.s., which we now briefly recall. Let k be any field, and let $(C^*(S; k)d)$ denote the graded differential algebra of k-valued singular cochains on S, augmented by a chosen nondegenerate base point. If $B(C^*(S))$ denotes the reduced bar construction on $C^*(S)$, we can then apply the Adams cobar construction to this and the result is the tensor algebra on $B^+(C^*(S))$, the canonical complement to k. The tensor powers $T^{\geq k}(B^+(C^*(S))$ are stable under the differential and upon filtering by them we obtain the M-M cohomology spectral sequence (E_r, d_r), where $E_2 \cong \mathrm{Ext}_{H_*(\Omega S; k)}(k, k)$ and which converges to $H^*(S; k)$ [FH,§9].

We need another, equivalent description. Let $(T(V), d) \xrightarrow{\simeq} (C^*(S, k), d)$ be a minimal free tensor model for the simply connected space S. Then $V = \bigoplus V_j$, $j \geq 2$, V is of finite type and the differential d increases "tensor degree" by at least one: $V \xrightarrow{d} T^{\geq 2}(V)$. Here, as above, $T^k|!(V)$ denotes $V \otimes \cdots \otimes V$ (k times). We bigrade $T(V)$ by setting $T(V)^{p,q} = [T^p(V)]^{p+q}$ where $p + q$ is the total (topological) degree. Above, the "$\simeq$" means that the morphism of differential algebras is a quism, i.e., it induces an isomorphism on cohomology.

Filtering $(T(V), d)$ by the ideal $(T^{\geq k}(V), d)$, we obtain a naively convergent spectral sequence (E_r, d_r) which coincides with the classical cohomology M-M s.s defined above from the E_l term on [H-L, Proposition 1.6]. From now on we refer to either as the M-M s.s.

Now let $(T(V), d) \xrightarrow{j} (T(V \oplus V_m), d') \xrightarrow{\simeq} (T(V)/T^{>m}(V), \overline{d})$ be a free model for the projection $(T(V), d) \xrightarrow{\pi} (T(V)/T^{>m}(V), \overline{d})$. Then $\mathrm{Mcat}(S; k)$ is defined as the least integer m for which there is a retraction of $T(V)$-differential modules $(T(V \oplus V_m), d') \xrightarrow{r} (T(V), d)$, i.e., r satisfies $r \circ j = \mathrm{id}$. (To define $\mathrm{Acat}(S; k)$, we ask that r be a map of algebras.)

The key ingredient of the proof of our theorem is a beautifully explicit model $(T(V) \otimes (k \oplus M), D) \xrightarrow{\Psi} (T(V)/T^{>m}(V), d)$ in [FHLT], which we describe. The bigrading of $T(V)$ above can be carried to $T(V)/T^{>m}(V)$ by making π homogeneous of bidegree $(0, 0)$. Let $M = M^{m,*} = sT^{m+1}(V)$, i.e., $M^{m,q} = s[T(V)^{m+1}, q]$. There is an isomorphism ω: $T(V) \otimes M \xrightarrow{\simeq} T^{>m}(V)$ defined by $\Omega(\Phi \otimes s\Psi) = (-1)^{|\Phi|}\Phi \cdot \Psi$ where $|\Phi|$ is the topological degree of Φ. We can then define a differential D on $T(V) \otimes (k \oplus M)$ by $D = d$ on $T(V) \otimes 1$ and $D = \omega - \omega^{-1} d\omega$ on $T(V) \otimes M$. This makes $(T(V) \otimes (k \oplus M)$, $D)$ a differential $T(V)$-module, and an easy calculation shows that the map

$$(T(V) \otimes (k \oplus M), D) \xrightarrow{\Psi} (T(V)/T^{>m}(V), \overline{d})$$

defined by $\Psi(v) = \pi(v)$, $\Psi(T(V) \otimes M) = 0$, is a quism.

We can filter $(T(V) \otimes (k \oplus M), D)$ by the left-hand degree and the E_1 term of the resulting spectral sequence has the form $(T(V) \otimes (k \oplus M), D_2)$ where $D_2 = d_2$ on $T(V)$ and $\omega - \omega^{-1} d_2 \omega$ on $T(V) \otimes M$; here d_2 denotes the quadratic part of the differential D_2: $V \to T^2(V)$. The map Ψ defined above remains a quism when considered as a map

$$(T(V) \otimes (k \oplus M), D_2) \xrightarrow{\Psi} (T(V)/T^{>m}(V), \overline{d}_2),$$

as can easily be checked. We note for future reference the observation that $\operatorname{Ker}\Psi = T^{>m}(V) \oplus T(V) \otimes M$ is acyclic for D and D_2 and that

$$T(V) \otimes (k \oplus M)^{\geq m+p,*} = T^{\geq m+p}(V) \oplus T^{\geq p}(V) \otimes M.$$

3. Proof of the theorem

To show that the M-M s.s. collapses at the $(m+1)$ st stage, it suffices to show that for each $k \geq 0$, if $a \in T(V)$ is such that $da \in T^{\geq m+k}(V)$ then there is an $\tilde{a} \in T^{\geq k}(V)$ with $d(a-\tilde{a}) = 0$. Note that for $k = 0$ or 1 we may choose $\tilde{a} = a$. Now assume $k \geq 2$ and let $(T(V), d) \to (C^*(S, k), d)$ be a minimal free tensor model as above. With $\mathrm{Mcat}(S; k) = m$, as in [FHLT, Step 6], this also guarantees a retraction r: $(T(V) \otimes (k \oplus M), D) \to (T(V), d)$.

Since $da \in \ker\Psi$ is a cocycle, there is some $b \in \ker\Psi$ with $Db = da$. Write $b = sb_0 + b_1 + b_2$ where $b_0 \in T^{m+1}(V)$, $b_1 \in T^{\geq m+1}(V)$, and $b_2 \in T^{\geq 1}(V) \otimes M$. A short computation shows that $Db - b_0 \in [T(V) \otimes (k \oplus M)]^{\geq m+2,*}$. Since $Db = da \in T^{\geq m+k}(V)$, this implies $b_0 = 0$. If $k = 2$, set $\tilde{a} = r(b)$. Then $d(a - \tilde{a}) = 0$ and $r(b) \in T^{\geq 2}(V)$ since $m \geq 1$, $r(M) \subseteq T^{\geq 1}(V)$ and r is a $T(V)$ module map.

If $k \geq 3$, write $b = b_{m+1} + b'$ where the filtration degree of b_{m+1} is $m+1$ and that of b' is at least $m+2$. Since $Db \in [T(V) \otimes (k \oplus M)]^{\geq m+k,*}$, we have $D_2 b_{m+1} = 0$. But $b_{m+1} \in \ker\Psi$ and so there is some u_m of filtration degree m with $D_2 u_m = b_{m+1}$. Noting that $D - D_2$ increases filtration degree by at least 2, we write $Db = D(Du_m - (D - D_2)u_m + b') = D(\tilde{b})$ where $\tilde{b}$ now has filtration degree at least $m+2$. As before, $r(\tilde{b}) \in T^{\geq 3}(V)$ and if $k = 3$, we set $\tilde{a} = r(\tilde{b})$ and stop.

For $k \geq 4$, continue in this manner, using the acyclicity of $\ker\Psi$ with respect to D_2 until we get to $D(b) = D(\tilde{b})$ where $\tilde{b}$ has filtration degree at least $m+k-1$. Then $r(\tilde{b}) \in T^{\geq k}(V)$ and as before, we set $a' = r(\tilde{b})$ to finish the proof.

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