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A NOTE ON A FIXED POINT THEOREM OF OKHEZIN

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ABSTRACT. In 1985 V. P. Okhezin proved that the cartesian product of a B-space X and a compact metric AR space has the fixed point property. In this paper it is shown that the cone over X and the suspension of X have the fixed point property.

1. INTRODUCTION

A *nest* is a monotone, increasing sequence of sets. A *B-space* X is an arcwise connected T_2-space such that every nest of arcs of X is contained in an arc of X. A space S has the *fixed point property* (abbreviated FPP) if for every continuous mapping f of S into itself, there exists a point p of S such that $f(p) = p$.

In 1946 Young [9] proved that every B-space has FPP. In 1954 Borsuk [1] showed that every arcwise connected, hereditarily unicoherent continuum X is a B-space and proved that X has FPP by a different method than Young's. In 1969 Holsztyński [3] generalized the above Young's result by a method similar to Borsuk's method. The"B-space" named by Holsztyński derives from Borsuk-Young arcwise connected space.

In 1985 Okhezin [5] verified that the cartesian product of a B-space and a compact metric AR space has FPP. In this note we shall prove the following theorem by his method.

Theorem. *Let X be a B-space and Y a compact metric AR space. Let F be a closed subset of Y, which may be empty. Define an equivalence relation $\sim$ in the cartesian product $X \times Y$ by $(x, y) \sim (x', y)$ for every $y \in F$ and any x, $x' \in X$. Then the quotient space $Q(X) = (X \times Y)/\sim$ has FPP.*

Corollary 1 [5]. *The cartesian product of a B-space and a compact metric AR space has FPP.*

Corollary 2. *The cone over a B-space has FPP.*

Corollary 3. *The suspension of a B-space has FPP.*

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2. Preliminary lemmas

From the definition of B-space we can easily get

Lemma 1. *A B-space X is uniquely arcwise connected, that is, for any two points x, y of X there is a unique arc in X with end points x, y.*

Definition 1 [8]. Let X be a connected T_1-space. A nonempty subset T of X is an *A-set* if $X \setminus T$ is the union of a collection $\{C\}$ of open sets each having a single point $q \in T$ as its boundary

$$X \setminus T = \bigcup C, \qquad \mathrm{Bd}\, C = q \in T.$$

If X is locally connected, then we may take all components of $X \setminus T$ as $\{C\}$.

Lemma 2. *Let X be a locally arcwise connected B-space. If T is an arcwise connected, closed subset of X then it is an A-set.*

Proof. Since T is closed and X is locally connected, every component C of $X \setminus T$ is open. Then Bd C consists of only one point. For, on the contrary, let x_1, x_2 be distinct points of Bd C. Since X is Hausdorff, there exist disjoint neighborhoods U_i of x_i $(i = 1, 2)$. Then a point $y_i \in U_i \cap C$ can be joined to x_i by an arc α_i in U_i. Let β be an arc in T joining x_1 to x_2 and γ an arc in C joining y_1 to y_2. Then $\alpha_1 \cup \beta \cup \alpha_2 \cup \gamma$ contains a simple closed curve, contrary to Lemma 1. □

Definition 2. Let T be an A-set of a connected T_1-space X. Then define a function $r\colon X \to T$ by

$$r(x) = \begin{cases} x & x \in T, \\ q = \mathrm{Bd}\, C & x \in C. \end{cases}$$

If X is locally connected, then r is a continuous mapping [8] called the *canonical retraction of X onto T*.

We easily have

Lemma 3. *Let X be a locally connected B-space, T an A-set of X, and $r\colon X \to T$ the canonical retraction of X onto T. Define $R\colon Q(X) \to Q(T)$ by $R(\langle x, y\rangle) = \langle r(x), y\rangle$, where $\langle x, y\rangle$ is the equivalence class of $(x, y) \in X \times Y$. Then R is a retraction of $Q(X)$ onto $Q(T)$, called the retraction associated with r.*

Definition 3. A *countable fan* is homeomorphic to the union of the closed line segments in the Euclidean plane R^2 joining the origin $(0, 0)$ to the point $(1, 1/n)$ $(n = 1, 2, \ldots)$. A *countable comb* is homeomorphic to the subspace C_0 of R^2 constructed as follows: Erect at $(1/n, 0)$ $(n = 1, 2, \ldots)$ in R^2 a perpendicular interval of length one. Then C_0 is the union of the erected intervals and the unit interval with end points $(0, 0)$, $(1, 0)$.

Lemma 4 (cf. [4, p. 84]). *Let X be a B-space and T an arcwise connected subset of X with infinitely many end points. Then T contains a countable fan or a countable comb.*

Proof. The arc in X with end points x, y is denoted by $[x, y]$. Define $(x, y) = [x, y] \setminus \{x, y\}$. Now pick a base point $a_1 \in T$. According to [4] we first introduce an order $\leqq$ in T by the rule: $x \leqq y$ if and only if $x \in [a_1, y]$.

For $x \in T$, $M(x) = \{y \in T | x \leqq y\}$ and $L(x) = \{y \in T | y \leqq x\}$. The notation $x \wedge y$ means the point $\sup\{L(x) \cap L(y)\}$. A *branch* B at x is a subset of $M(x)$ that is maximal with respect to the property: If y, $z \in B \setminus \{x\}$ then $y \wedge z \in B \setminus \{x\}$. Note that every maximal element of T is an end point of T and the only other possible end point of T is a_1.

Assume that T contains neither countable fan nor countable comb. Then there are only finitely many branches at a_1. For if not, there would be a countable fan in T. Hence a branch B_1 at a_1 has infinitely many end points. Let m_1 be one of these end points. The arc $[a_1, m_1]$ contains only finitely many branch points. For if not, there would exist a countable comb.

Inductively we can construct three sequences $\{m_i\}$, $\{a_i\}$, and $\{B_i\}$ with the following properties:

(i) B_i is a branch at the point a_i with infinitely many end points.
(ii) m_i is an end point of B_i.
(iii) $a_{i+1} \in (a_i, m_i)$.

Denote the arc $\bigcup_{i=2}^{n}[a_i, a_{i+1}]$ by A_n $(n \geqq 2)$. Since X is a B-space, the nest $\{A_n : n \geqq 1\}$ of arcs is contained in an arc of X. Let $\lim a_i = a$. Then the set $(\bigcup_{n=2}^{\infty} A_n) \cup (\bigcup_{i=1}^{\infty}[a_{i+1}, m_i]) \cup \{a\}$ is a countable comb, which is a contradiction. Thus X contains a countable fan or a countable comb. □

Lemma 5. *Let X, Y, and F be as in Theorem. Let T be a finite tree in X and define $Q(T)$ in the same way as $Q(X)$. Then $Q(T)$ has FPP.*

Proof. If $F = Y$ then $Q(T)$ is homeomorphic to Y, and hence $Q(T)$ has FPP. If $F = \phi$ then $Q(T) = T \times Y$, and hence $Q(T)$ is a compact metric AR space. Therefore $Q(T)$ has FPP.

For the other case, suppose that T is contained in the unit disk D of the Euclidean plane. Let ψ be a retraction of D onto T. We may assume that $\max d(y, F) \leqq 1$ for $y \in Y$, where d is a metric on Y. Then the set $H = \{(d(y, F) \cdot x, y)(x, y) \in D \times Y\}$ is homeomorphic to $Q(D)$, which is defined in the same way as $Q(X)$, and the subset $K = \{(d(y, F) \cdot x, y)(x, y) \in T \times Y\}$ of H is homeomorphic to $Q(T)$. Obviously H is a retract of $D \times Y$. Since $D \times Y$ has FPP, so does H. Furthermore a retraction $g: H \to K$ is defined by

$$g((x, y)) = \begin{cases} (d(y, F) \cdot \psi(x/d(y, F)), y) & d(y, F) \neq 0, \\ (x, y) & d(y, F) = 0. \end{cases}$$

Hence K has FPP and so does $Q(T)$. □

3. Key lemma

Lemma 6. *Let X be a locally arcwise connected B-space and $Q(X)$ the same as in Theorem. Then $Q(X)$ has FPP.*

Proof. Let K be the set $\{\langle x, y\rangle \in Q(X) y \in F\}$. Then $Q(X) \setminus K$ is homeomorphic to $X \times (Y \setminus F)$. Let $\pi: Q(X) \setminus K \to X$ and $p: Q(X) \to Y$ be the projections defined by $\pi(\langle x, y\rangle) = x$ and $p(\langle x, y\rangle) = y$, respectively.

Now let $f: Q(X) \to Q(X)$ be any continuous mapping of $Q(X)$ into itself. On the contrary, suppose that f has no fixed point. To lead to a contradiction, we shall divide our argument into four steps.

Step 1. We construct a sequence $\{z_n\}$ of points in $Q(X)$ and a nest $\{T_n\}$ of finite trees in X satisfying the following four conditions.

(3.1) $R_n f(z_n) = z_n$ $(n \geqq 1)$, where $R_n\colon Q(X) \to Q(T_{n-1})$ is the retraction associated with the canonical retraction $r_n\colon X \to T_{n-1}$, and neither z_n nor $f(z_n)$ belong to K.

(3.2) $\pi(z_n) \in T_{n-1}$ and $\pi f(z_n) \notin T_{n-1}$ $(n \geqq 1)$.

(3.3) $T_{n-1} \cap [\pi(z_n), \pi f(z_n)] = \{\pi(z_n)\}$.

(3.4) $T_n = \bigcup_{i=1}^{n} [\pi(z_i), \pi f(z_i)]$.

The sequences $\{z_n\}$ and $\{T_n\}$ are inductively obtained in the following manner.

Let T_0 be any point of X. Since $Q(T_0)$ is homeomorphic to Y, it has FPP. By Lemma 3, the restriction $R_1 f|Q(T_0)$ is continuous. Therefore there exists a point $z_1 \in Q(T_0)$ such that $R_1 f(z_1) = z_1$. By our assumption that f has no fixed point, neither z_1 nor $f(z_1)$ belong to K. Note that $\pi(z_1) = T_0$ and put $T_1 = [\pi(z_1), \pi f(z_1)]$.

For positive integer n, since $Q(T_{n-1})$ has FPP by Lemma 5 and $R_n f|Q(T_{n-1})\colon Q(T_{n-1}) \to Q(T_{n-1})$ is continuous, there exists a point $z_n \in Q(T_{n-1})$ such that $R_n f(z_n) = z_n$. Since f has no fixed point by our assumption, neither z_n nor $f(z_n)$ are in K.

Clearly $\pi(z_n) \in T_{n-1}$. However $\pi f(z_n) \notin T_{n-1}$. For if not, $f(z_n) \in Q(T_{n-1})$ and $R_n f(z_n) = f(z_n)$ because $R_n\colon Q(X) \to Q(T_{n-1})$ is a retraction. Therefore $f(z_n) = z_n$, a contradiction. From (3.1) it follows that $r_n \pi f(z_n) = \pi(z_n)$. Hence by the definition of r_n we have $T_{n-1} \cap [\pi(z_n), \pi f(z_n)] = \pi(z_n)$. Put $T_n = T_{n-1} \cup [\pi(z_n), \pi f(z_n)]$.

Step 2. By our construction, the set $T = \bigcup_{n=1}^{\infty} T_n$ has the following four properties.

(3.5) For $n < m$, $[\pi(z_n), \pi f(z_n)] \cap [\pi(z_m), \pi f(z_m)]$ is empty or $\{\pi(z_m)\}$.

(3.6) If e is an end point of T then $e = \pi(z_1)$ or $e = \pi f(z_n)$ for some integer n.

(3.7) If b is a branch point of T then $b = \pi(z_n)$ for some integer n.

(3.8) If b is a branch point of T whose order is infinite, then there is an increasing sequence of integers $n_1 < n_2 < \cdots$ such that $\pi(z_{n_i}) = b$.

Step 3. If a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ converges to a point z, then (3.9) and (3.10) hold.

(3.9) $pf(z) = p(z)$ and neither z nor $f(z)$ are in K. For, $pf(z_{n_i}) = p(z_{n_i})$ by (3.1) and hence $pf(z) = \lim pf(z_{n_i}) = \lim p(z_{n_i}) = p(z)$. Suppose, on the contrary, that $z \in K$. Then $pf(z) = p(z) \in F$. Therefore $f(z) = z$, a contradiction. Similarly $f(z)$ is not in K.

(3.10) There exists no arc α of X such that $\pi(z_{n_i}) \in \alpha \cap T$ $(i \geqq 1)$.

Suppose, on the contrary, that there exists such an arc α. By (3.9) there exist $\pi(z)$ and $\pi f(z)$. Since $pf(z) = p(z)$ and $f(z) \neq z$, it holds that $\pi f(z) \neq \pi(z)$. Since X is Hausdorff, there exist disjoint neighborhoods U, V of $\pi f(z)$, $\pi(z)$ in X, respectively. Let U_1 be a neighborhood of $\pi f(z)$ such that $U_1 \subset U$ and each pair of points in U_1 can be joined by an arc in U. Since the map πf is continuous, we can find a neighborhood M of $\pi(z)$ in α and a neighborhood N of $p(z)$ in $Y \backslash F$ such that $\pi f(M \times N) \subset U_1$. Let z_{n_k}, z_{n_l} be in $M \times N$. Then $\pi(z_{n_k})$, $\pi(z_{n_l})$ are in M and $\pi f(z_{n_k})$, $\pi f(z_{n_l})$ are distinct points in $\pi f(M \times N)$ by (3.2) and (3.3). Let β be an arc in U joining $\pi f(z_{n_k})$ to $\pi f(z_{n_l})$. The set $[\pi(z_{n_k}), \pi f(z_{n_l})] \cup \beta \cup [\pi f(z_{n_l}), \pi(z_{n_l})] \cup [\pi(z_{n_l}), \pi(z_{n_l})]$ contains a simple closed curve in X, which contradicts Lemma 1.

Step 4. If the end points of T are finite in number, by the construction of T there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $\{[\pi(z_{n_1}), \pi(z_{n_i})], i > 1\}$ is a nest of arcs. Let α be an arc of X containing this nest. We may suppose that $\{\pi(z_{n_i})\}$ and $\{p(z_{n_i})\}$ converge to $a \in X$ and $y \in Y$, respectively. Put $(a, y) = z$. Then $\lim z_{n_i} = z$. This contradicts (3.10).

If the end points of T are infinite in number, by Lemma 4 there exists a countable fan or a countable comb in T. For a countable fan with branch point b, by (3.8) there is an increasing sequence of integers $n_1 < n_2 < \cdots$ such that $\pi(z_{n_i}) = b$. We may suppose that $\{p(z_{n_i})\}$ converges to a point $y \in Y$. Then $\lim z_{n_i} = (b, y)$. The point b is contained in an arc of X, contrary to (3.10).

For a countable comb $C \subset T$, let $h: C_0 \to C$ be a homeomorphism of C_0 onto C, where C_0 is the subset of R^2 in Definition 3. Let $a = h((0, 0))$. By (3.7) we may put $\pi(z_{n_k}) = h((1/k, 0))$ and assume that $\{p(z_{n_k})\}$ converges to $y \in Y$. Then $\lim z_{n_k} = (a, y)$. The sequence $\{\pi(z_{n_k})\}$ is contained in the arc $\alpha = [a, \pi(z_{n_i})]$ of X, which violates (3.10). □

4. Proof of Theorem

Definition 4. Let S be a subset of an arcwise connected space. An *arc component* of S is a maximal arcwise connected subset of S. Young's arc topology [9, 6] is the topology with the arc components of open sets of the given topology as basis.

Let (X, τ) be a B-space with topology τ and $Q(X, \tau)$ the quotient space in the theorem. Let $f: Q(X, \tau) \to Q(X, \tau)$ be any continuous mapping and λ be Young's arc topology on X. Then (X, λ) is a locally arcwise connected T_2-space. The mapping $f: Q(X, \lambda) \to Q(X, \tau)$ is continuous, because λ is finer than τ. Since $Q(X, \tau)$ is Hausdorff, the image of any arcwise connected subset under $f: Q(X, \lambda) \to Q(X, \tau)$ is arcwise connected. Hence the mapping $f: Q(X, \lambda) \to Q(X, \lambda)$ is continuous and has a fixed point by Lemma 6. Thus $Q(X, \tau)$ has FPP.

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