A NOTE ON A FIXED POINT THEOREM OF OKHEZIN

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Abstract. In 1985 V. P. Okhezin proved that the cartesian product of a $B$-space $X$ and a compact metric AR space has the fixed point property. In this paper it is shown that the cone over $X$ and the suspension of $X$ have the fixed point property.

1. Introduction

A nest is a monotone, increasing sequence of sets. A $B$-space $X$ is an arcwise connected $T_2$-space such that every nest of arcs of $X$ is contained in an arc of $X$. A space $S$ has the fixed point property (abbreviated FPP) if for every continuous mapping $f$ of $S$ into itself, there exists a point $p$ of $S$ such that $f(p) = p$.

In 1946 Young [9] proved that every $B$-space has FPP. In 1954 Borsuk [1] showed that every arcwise connected, hereditarily unicoherent continuum $X$ is a $B$-space and proved that $X$ has FPP by a different method than Young's. In 1969 Holsztyński [3] generalized the above Young's result by a method similar to Borsuk's method. The "$B$-space" named by Holsztyński derives from Borsuk-Young arcwise connected space.

In 1985 Okhezin [5] verified that the cartesian product of a $B$-space and a compact metric AR space has FPP. In this note we shall prove the following theorem by his method.

Theorem. Let $X$ be a $B$-space and $Y$ a compact metric AR space. Let $F$ be a closed subset of $Y$, which may be empty. Define an equivalence relation $\sim$ in the cartesian product $X \times Y$ by $(x, y) \sim (x', y)$ for every $y \in F$ and any $x, x' \in X$. Then the quotient space $Q(X) = (X \times Y) / \sim$ has FPP.

Corollary 1 [5]. The cartesian product of a $B$-space and a compact metric AR space has FPP.

Corollary 2. The cone over a $B$-space has FPP.

Corollary 3. The suspension of a $B$-space has FPP.
2. PRELIMINARY LEMMAS

From the definition of $B$-space we can easily get

**Lemma 1.** A $B$-space $X$ is uniquely arcwise connected, that is, for any two points $x, y$ of $X$ there is a unique arc in $X$ with end points $x, y$.

**Definition 1** [8]. Let $X$ be a connected $T_1$-space. A nonempty subset $T$ of $X$ is an $A$-set if $X \setminus T$ is the union of a collection $\{C\}$ of open sets each having a single point $q \in T$ as its boundary

\[ X \setminus T = \bigcup C, \quad \text{Bd } C = q \in T. \]

If $X$ is locally connected, then we may take all components of $X \setminus T$ as $\{C\}$.

**Lemma 2.** Let $X$ be a locally arcwise connected $B$-space. If $T$ is an arcwise connected, closed subset of $X$ then it is an $A$-set.

**Proof.** Since $T$ is closed and $X$ is locally connected, every component $C$ of $X \setminus T$ is open. Then $\text{Bd } C$ consists of only one point. For, on the contrary, let $x_1, x_2$ be distinct points of $\text{Bd } C$. Since $X$ is Hausdorff, there exist disjoint neighborhoods $U_i$ of $x_i$ ($i = 1, 2$). Then a point $y_i \in U_i \cap C$ can be joined to $x_i$ by an arc $\alpha_i$ in $U_i$. Let $\beta$ be an arc in $T$ joining $x_1$ to $x_2$ and $\gamma$ an arc in $C$ joining $y_1$ to $y_2$. Then $\alpha_1 \cup \beta \cup \alpha_2 \cup \gamma$ contains a simple closed curve, contrary to Lemma 1. $\Box$

**Definition 2.** Let $T$ be an $A$-set of a connected $T_1$-space $X$. Then define a function $r : X \to T$ by

\[ r(x) = \begin{cases} x & x \in T, \\ q = \text{Bd } C & x \in C. \end{cases} \]

If $X$ is locally connected, then $r$ is a continuous mapping [8] called the canonical retraction of $X$ onto $T$.

We easily have

**Lemma 3.** Let $X$ be a locally connected $B$-space, $T$ an $A$-set of $X$, and $r : X \to T$ the canonical retraction of $X$ onto $T$. Define $R : \mathcal{Q}(X) \to \mathcal{Q}(T)$ by $R((x, y)) = (r(x), y)$, where $(x, y)$ is the equivalence class of $(x, y) \in X \times Y$. Then $R$ is a retraction of $\mathcal{Q}(X)$ onto $\mathcal{Q}(T)$, called the retraction associated with $r$.

**Definition 3.** A countable fan is homeomorphic to the union of the closed line segments in the Euclidean plane $\mathbb{R}^2$ joining the origin $(0, 0)$ to the point $(1, 1/n)$ ($n = 1, 2, \ldots$). A countable comb is homeomorphic to the subspace $C_0$ of $\mathbb{R}^2$ constructed as follows: Erect at $(1/n, 0)$ ($n = 1, 2, \ldots$) in $\mathbb{R}^2$ a perpendicular interval of length one. Then $C_0$ is the union of the erected intervals and the unit interval with end points $(0, 0), (1, 0)$.

**Lemma 4** (cf. [4, p. 84]). Let $X$ be a $B$-space and $T$ an arcwise connected subset of $X$ with infinitely many end points. Then $T$ contains a countable fan or a countable comb.

**Proof.** The arc in $X$ with end points $x, y$ is denoted by $[x, y]$. Define $(x, y) = [x, y] \setminus \{x, y\}$. Now pick a base point $a_1 \in T$. According to [4] we first introduce an order $\preceq$ in $T$ by the rule: $x \preceq y$ if and only if $x \in [a_1, y]$.
For $x \in T$, $M(x) = \{ y \in T | x \leq y \}$ and $L(x) = \{ y \in T | y \leq x \}$. The notation $x \land y$ means the point $\sup \{ L(x) \cap L(y) \}$. A branch $B$ at $x$ is a subset of $M(x)$ that is maximal with respect to the property: If $y, z \in B \setminus \{ x \}$ then $y \land z \in B \setminus \{ x \}$. Note that every maximal element of $T$ is an end point of $T$ and the only other possible end point of $T$ is $a_1$.

Assume that $T$ contains neither countable fan nor countable comb. Then there are only finitely many branches at $a_1$. For if not, there would be a countable fan in $T$. Hence a branch $B_1$ at $a_1$ has infinitely many end points. Let $m_1$ be one of these end points. The arc $[a_1, m_1]$ contains only finitely many branch points. For if not, there would exist a countable comb.

Inductively we can construct three sequences $\{m_i\}$, $\{a_i\}$, and $\{B_i\}$ with the following properties:

1. $B_i$ is a branch at the point $a_i$ with infinitely many end points.
2. $m_i$ is an end point of $B_i$.
3. $a_{i+1} \in \{a_i, m_i\}$.

Denote the arc $\bigcup_{i=2}^{n}[a_i, a_{i+1}]$ by $A_n$ $(n \geq 2)$. Since $X$ is a $B$-space, the nest $\{A_n : n \geq 1\}$ of arcs is contained in an arc of $X$. Let $\lim a_i = a$. Then the set $(U_{n=2}^{\infty}A_n) \cup (\bigcup_{i=1}^{\infty}[a_{i+1}, m_i]) \cup \{a\}$ is a countable comb, which is a contradiction. Thus $X$ contains a countable fan or a countable comb. □

Lemma 5. Let $X$, $Y$, and $F$ be as in Theorem. Let $T$ be a finite tree in $X$ and define $Q(T)$ in the same way as $Q(X)$. Then $Q(T)$ has FPP.

Proof. If $F = Y$ then $Q(T)$ is homeomorphic to $Y$, and hence $Q(T)$ has FPP. If $F = \phi$ then $Q(T) = T \times Y$, and hence $Q(T)$ is a compact metric AR space. Therefore $Q(T)$ has FPP.

For the other case, suppose that $T$ is contained in the unit disk $D$ of the Euclidean plane. Let $\psi$ be a retraction of $D$ onto $T$. We may assume that $\max d(y, F) \leq 1$ for $y \in Y$, where $d$ is a metric on $Y$. Then the set $H = \{(d(y, F) \cdot x, y)(x, y) \in D \times Y\}$ is homeomorphic to $Q(D)$, which is defined in the same way as $Q(X)$, and the subset $K = \{(d(y, F) \cdot x, y)(x, y) \in T \times Y\}$ of $H$ is homeomorphic to $Q(T)$. Obviously $H$ is a retract of $D \times Y$. Since $D \times Y$ has FPP, so does $H$. Furthermore a retraction $g : H \rightarrow K$ is defined by

$$g((x, y)) = \begin{cases} (d(y, F) \cdot \psi(x/d(y, F)), y) & d(y, F) \neq 0, \\ (x, y) & d(y, F) = 0. \end{cases}$$

Hence $K$ has FPP and so does $Q(T)$. □

3. Key lemma

Lemma 6. Let $X$ be a locally arcwise connected $B$-space and $Q(X)$ the same as in Theorem. Then $Q(X)$ has FPP.

Proof. Let $K$ be the set $\{(x, y) \in Q(X) | y \in F\}$. Then $Q(X) \setminus K$ is homeomorphic to $X \times (Y \setminus F)$. Let $\pi : Q(X) \setminus K \rightarrow X$ and $p : Q(X) \rightarrow Y$ be the projections defined by $\pi((x, y)) = x$ and $p((x, y)) = y$, respectively.

Now let $f : Q(X) \rightarrow Q(X)$ be any continuous mapping of $Q(X)$ into itself. On the contrary, suppose that $f$ has no fixed point. To lead to a contradiction, we shall divide our argument into four steps.

Step 1. We construct a sequence $\{z_n\}$ of points in $Q(X)$ and a nest $\{T_n\}$ of finite trees in $X$ satisfying the following four conditions.
(3.1) \( R_n f(z_n) = z_n \quad (n \geq 1) \), where \( R_n : Q(X) \to Q(T_{n-1}) \) is the retraction associated with the canonical retraction \( r_n : X \to T_{n-1} \), and neither \( z_n \) nor \( f(z_n) \) belong to \( K \).

(3.2) \( \pi(z_n) \in T_{n-1} \) and \( \pi(f(z_n)) \notin T_{n-1} \quad (n \geq 1) \).

(3.3) \( T_{n-1} \cap [\pi(z_n), \pi(f(z_n))] = \{\pi(z_n)\} \).

(3.4) \( T_n = \bigcup_{i=1}^{n} [\pi(z_i), \pi(f(z_i))] \).

The sequences \( \{z_n\} \) and \( \{T_n\} \) are inductively obtained in the following manner.

Let \( T_0 \) be any point of \( X \). Since \( Q(T_0) \) is homeomorphic to \( Y \), it has FPP. By Lemma 3, the restriction \( R_1 f|Q(T_0) \) is continuous. Therefore there exists a point \( z_1 \in Q(T_0) \) such that \( R_1 f(z_1) = z_1 \). By our assumption that \( f \) has no fixed point, neither \( z_1 \) nor \( f(z_1) \) belong to \( K \). Note that \( \pi(z_1) = T_0 \) and put \( T_1 = [\pi(z_1), \pi(f(z_1))] \).

For positive integer \( n \), since \( Q(T_{n-1}) \) has FPP by Lemma 5 and \( R_n f|Q(T_{n-1}) : Q(T_{n-1}) \to Q(T_{n-1}) \) is continuous, there exists a point \( z_n \in Q(T_{n-1}) \) such that \( R_n f(z_n) = z_n \). Since \( f \) has no fixed point by our assumption, neither \( z_n \) nor \( f(z_n) \) are in \( K \).

Clearly \( \pi(z_n) \in T_{n-1} \). However \( \pi(f(z_n)) \notin T_{n-1} \). For if not, \( f(z_n) \in Q(T_{n-1}) \) and \( R_n f(z_n) = f(z_n) \) because \( R_n : Q(X) \to Q(T_{n-1}) \) is a retraction. Therefore \( f(z_n) = z_n \), a contradiction. From (3.1) it follows that \( r_n \pi(f(z_n)) = \pi(z_n) \). Hence by the definition of \( r_n \) we have \( T_{n-1} \cap [\pi(z_n), \pi(f(z_n))] = \{\pi(z_n)\} \).

Put \( T_0 = T_{n-1} \cup [\pi(z_n), \pi(f(z_n))] \).

Step 2. By our construction, the set \( T = \bigcup_{n=1}^{\infty} T_n \) has the following four properties.

(3.5) For \( n < m \), \( [\pi(z_n), \pi(f(z_n))] \cap [\pi(z_m), \pi(f(z_m))] \) is empty or \( \{\pi(z_m)\} \).

(3.6) If \( e \) is an end point of \( T \) then \( e = \pi(z_1) \) or \( e = \pi(f(z_n)) \) for some integer \( n \).

(3.7) If \( b \) is a branch point of \( T \) then \( b = \pi(z_n) \) for some integer \( n \).

(3.8) If \( b \) is a branch point of \( T \) whose order is infinite, then there is an increasing sequence of integers \( n_1 < n_2 < \cdots \) such that \( \pi(z_n) = b \).

Step 3. If a subsequence \( \{z_{n_k}\} \) of \( \{z_n\} \) converges to a point \( z \), then (3.9) and (3.10) hold.

(3.9) \( pf(z) = p(z) \) and neither \( z \) nor \( f(z) \) are in \( K \). For, \( pf(z_{n_k}) = p(z_{n_k}) \) by (3.1) and hence \( pf(z) = \lim pf(z_{n_k}) = \lim p(z_{n_k}) = p(z) \). Suppose, on the contrary, that \( z \in K \). Then \( pf(z) = p(z) \in F \). Therefore \( f(z) = z \), a contradiction. Similarly \( f(z) \) is not in \( K \).

(3.10) There exists no arc \( \alpha \) of \( X \) such that \( \pi(z_{n_k}) \in \alpha \cap T \quad (i \geq 1) \).

Suppose, on the contrary, that there exists such an arc \( \alpha \). By (3.9) there exist \( \pi(z) \) and \( pf(z) \). Since \( p f(z) = p(z) \) and \( f(z) \neq z \), it holds that \( \pi f(z) \neq \pi(z) \). Since \( X \) is Hausdorff, there exist disjoint neighborhoods \( U, V \) of \( pf(z) \), \( \pi(z) \) in \( X \), respectively. Let \( U_1 \) be a neighborhood of \( \pi f(z) \) such that \( U_1 \subset U \) and each pair of points in \( U_1 \) can be joined by an arc in \( U \). Since the map \( \pi f \) is continuous, we can find a neighborhood \( M \) of \( \pi(z) \) in \( \alpha \) and a neighborhood \( N \) of \( p(z) \) in \( Y \setminus F \) such that \( \pi f(M \times N) \subset U_1 \). Let \( z_{n_k}, z_{n_k}, z_{n_k} \) be in \( M \times N \). Then \( \pi(z_{n_k}), \pi(z_{n_k}), \pi(z_{n_k}) \) are in \( M \) and \( \pi f(z_{n_k}), \pi f(z_{n_k}) \) are distinct points in \( \pi f(M \times N) \) by (3.2) and (3.3). Let \( \beta \) be an arc in \( U \) joining \( \pi f(z_{n_k}) \) to \( \pi f(z_{n_k}) \). The set \( \{\pi(z_{n_k}), \pi f(z_{n_k})\} \cup \beta \cup \{\pi f(z_{n_k}), \pi f(z_{n_k})\} \) contains a simple closed curve in \( X \), which contradicts Lemma 1.
Step 4. If the end points of $T$ are finite in number, by the construction of $T$ there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $[\pi(z_{n_i}), \pi(z_{n_{i+1}})]$, $i > 1$ is a nest of arcs. Let $\alpha$ be an arc of $X$ containing this nest. We may suppose that $\{\pi(z_{n_i})\}$ and $\{\rho(z_{n_i})\}$ converge to $a \in X$ and $y \in Y$, respectively. Put $(a, y) = z$. Then $\lim z_{n_i} = z$. This contradicts (3.10).

If the end points of $T$ are infinite in number, by Lemma 4 there exists a countable fan or a countable comb in $T$. For a countable fan with branch point $b$, by (3.8) there is an increasing sequence of integers $n_1 < n_2 < \cdots$ such that $\pi(z_{n_i}) = b$. We may suppose that $\{\rho(z_{n_i})\}$ converges to a point $y \in Y$. Then $\lim z_{n_i} = (b, y)$. The point $b$ is contained in an arc of $X$, contrary to (3.10).

For a countable comb $C \subset T$, let $h: C_0 \to C$ be a homeomorphism of $C_0$ onto $C$, where $C_0$ is the subset of $R^2$ in Definition 3. Let $a = h((0, 0))$. By (3.7) we may put $\pi(z_{n_i}) = h((1/k, 0))$ and assume that $\{\rho(z_{n_i})\}$ converges to $y \in Y$. Then $\lim z_{n_i} = (a, y)$. The sequence $\{\pi(z_{n_i})\}$ is contained in the arc $\alpha = [a, \pi(z_{n_i})]$ of $X$, which violates (3.10). \(\square\)

4. Proof of Theorem

Definition 4. Let $S$ be a subset of an arcwise connected space. An arc component of $S$ is a maximal arcwise connected subset of $S$. Young's arc topology [9, 6] is the topology with the arc components of open sets of the given topology as basis.

Let $(X, \tau)$ be a $B$-space with topology $\tau$ and $Q(X, \tau)$ the quotient space in the theorem. Let $f: Q(X, \tau) \to Q(X, \tau)$ be any continuous mapping and $\lambda$ be Young's arc topology on $X$. Then $(X, \lambda)$ is a locally arcwise connected $T_2$-space. The mapping $f: Q(X, \lambda) \to Q(X, \tau)$ is continuous, because $\lambda$ is finer than $\tau$. Since $Q(X, \lambda)$ is Hausdorff, the image of any arcwise connected subset under $f: Q(X, \lambda) \to Q(X, \tau)$ is arcwise connected. Hence the mapping $f: Q(X, \lambda) \to Q(X, \lambda)$ is continuous and has a fixed point by Lemma 6. Thus $Q(X, \tau)$ has FPP.

REFERENCES


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