

THE DUAL OF A GÂTEAUX SMOOTH BANACH SPACE IS WEAK STAR FRAGMENTABLE

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(Communicated by William J. Davis)

ABSTRACT. The techniques of Preiss-Phelps-Namioka is used to prove that if a Banach space E admits a Gâteaux smooth norm then its dual E^* , endowed the weak star topology, is fragmentable.

0. INTRODUCTION

The main aim of this paper is to prove the title statement and to establish further relations between the theory of weak Asplund spaces and the notion of fragmentability. The latter notion was introduced by Jayne and Rogers in [JR].

0.1. Definition. Let X be a topological space and ρ be a metric on it. We say that ρ fragments X if for every $\varepsilon > 0$ and for every nonempty subset Y of X there exists a nonempty relatively open subset U of Y whose ρ -diameter is not greater than ε . The topological space X is said to be fragmentable if there exists a fragmenting metric on it.

0.2. Definition. The Banach space E is called weak Asplund if every convex function defined on an open convex subset of E is Gâteaux differentiable at each point of a dense G_δ subset of its domain.

For additional information about weak Asplund spaces the reader is referred to the book of R. Phelps [Ph].

0.3. Theorem of Asplund. *If the dual space E^* admits a dual strictly convex norm then E is weak Asplund.*

Since, in this case, the corresponding norm in E is Gâteaux differentiable (off zero), it was natural to ask if the assertion of the above theorem of Asplund is valid for every Banach space E with a norm which is differentiable at each nonzero point (i.e., a Gâteaux smooth norm). This long standing question was recently answered positively in [PPhN]:

0.4. Theorem of Preiss-Phelps-Namioka. *If the norm of E is Gâteaux smooth then E is weak Asplund.*

Received by the editors August 12, 1990.

1991 *Mathematics Subject Classification.* Primary 46B20, 54C60.

Key words and phrases. Weak Asplund spaces, fragmentability.

The connection between the notions "fragmentability" and "weak Asplund" is revealed by the following statement, which was proved in [ChK].

0.5. Theorem. *If E^* with its weak-star topology is fragmentable then E is weak Asplund.*

It was not clear, however, whether this result implies the above mentioned theorems of Asplund and Preiss-Phelps-Namioka, respectively. In [R1] the author has proved that this is the case for the theorem of Asplund. Combining this with the techniques developed in [PPhN] we show in §1 that the Preiss-Phelps-Namioka theorem can also be considered as a corollary of Theorem 0.5. (In fact, our main result is a further step after Theorems 1.6 and 2.2 of [PPhN] and its proof uses the same construction.)

Our approach needs no "external notions" like games and mappings. It provides additional information about the intrinsic structure of (E^*, w^*) .

The results in the last section generalize the theorems from §3 of [PPhN].

The proofs in this paper make use of a characterization of fragmentable spaces by the following notion.

0.6. Definition. A well-ordered family $\mathfrak{U} = \{U_\xi : 0 \leq \xi < \xi_0\}$ of subsets of the topological space X is said to be a relatively open partitioning of X if

- (i) $U_0 = \emptyset$;
- (ii) U_ξ is contained in $X \setminus (\bigcup_{\eta < \xi} U_\eta)$ and is relatively open in it for every ξ , $0 < \xi < \xi_0$;
- (iii) $X = (\bigcup_{\xi < \xi_0} U_\xi)$.

A family \mathfrak{U} of subsets of the topological space X is said to be a σ -relatively open partitioning of X if $\mathfrak{U} = \bigcup_{n=1}^{\infty} \mathfrak{U}^n$, where \mathfrak{U}^n , $n = 1, 2, \dots$ are relatively open partitionings of X . \mathfrak{U} is said to separate the points of X if whenever x and y are two different elements of X there exists n such that x and y belong to different elements of the partitioning \mathfrak{U}^n . In this case we say that X admits a separating σ -relatively open partitioning.

A topological space X is fragmentable if and only if it admits a separating σ -relatively open partitioning (see [R, Theorem 1.9]).

1. DUALS OF GÂTEAUX SMOOTH SPACES

1.1. Theorem. *Let E be a Banach space and $\|\cdot\|$ be a Gâteaux smooth norm on it. Then the topological space (E^*, w^*) is fragmentable.*

Proof. Step 1. Let us denote by B^* the dual unit ball. As $E^* = \bigcup_{m=1}^{\infty} mB^*$, it suffices to prove that (B^*, w^*) is a fragmentable space.

Step 2. Basic step of the construction.

Let U be a subset of B^* , p be an equivalent norm on E , and ε, β be two positive numbers. Then we construct a relatively open partitioning $\mathfrak{B} = \{V_\xi : 0 \leq \xi < \xi_0\}$ of (U, w^*) and associate to every element V of \mathfrak{B} a nonnegative real number s_V , an element e_V of E with $p(e_V) = 1$, and a new equivalent norm p_V on E in the following way.

We put $V_0 = \emptyset$, $s_{V_0} = 0$, and e_{V_0}, p_{V_0} be arbitrary.

If we have constructed V_η for every ordinal number $\eta < \xi$, we consider the set $R_\xi = U \setminus (\bigcup_{\eta < \xi} V_\eta)$. If $R_\xi = \emptyset$, we put $\xi_0 = \xi$ and finish the process.

If not, we put $s_{V_\xi} = \sup\{p^*(x^*) : x^* \in R_\xi\}$ where p^* is the norm conjugate to p .

If $s_{V_\xi} = 0$, it means that $R_\xi = \{0\}$. Then we put $V_\xi = \{0\}$, $\xi_0 = \xi + 1$, and e_{V_ξ} , p_{V_ξ} be arbitrary and finish again.

If $s_{V_\xi} > 0$, there exists an element e_{V_ξ} of E with $p(e_{V_\xi}) = 1$ such that the set

$$V_\xi = \{x^* \in R_\xi : \langle x^*, e_{V_\xi} \rangle > (1 - \varepsilon)s_{V_\xi}\}$$

is nonempty. Moreover, V_ξ is w^* -relatively open in R_ξ . We put $q_{V_\xi}(x) = \inf\{\|x - \lambda e_{V_\xi}\| : \lambda \in \mathbb{R}\}$ for every $x \in E$ and $p_{V_\xi}^2 = p^2 + \beta \cdot q_{V_\xi}$. Thus the relatively open partitioning \mathfrak{B} is constructed.

Step 3. Construction of the σ -relatively open partitioning $\mathfrak{U} = \bigcup_{n=1}^\infty \mathfrak{U}^n$ of (B^*, w^*) . Following the construction in [PPhN] we fix two sequences of positive numbers $1/2 > \varepsilon_1 > \varepsilon_2 > \dots$ and $\beta_1 > \beta_2 > \dots$ such that $\varepsilon_k \rightarrow 0$, $\sum \beta_k^2 < 3$, and $\sum (\varepsilon_k)^{1/2} / \beta_k < \infty$.

Applying the basic step with $U := B^*$, $p := \|\cdot\|$, $\varepsilon := \varepsilon_1$, and $\beta := \beta_1$, we obtain a relatively open partitioning $\mathfrak{U}^1 = \{U_\xi^1 : 0 \leq \xi < \xi_1\}$ of (B^*, w^*) , and for every element $U \in \mathfrak{U}^1$ we get $s_U \geq 0$, $e_U \in E$ with $\|e_U\| = 1$ and an equivalent norm p_U on E .

If we have constructed the relatively open partitionings $\mathfrak{U}^1, \mathfrak{U}^2, \dots, \mathfrak{U}^n$, we consider a fixed element U_ξ^n of $\mathfrak{U}^n = \{U_\xi^n : 0 \leq \xi < \xi_n\}$. Then we apply the basic step to U_ξ^n with $p = p_{U_\xi^n}$, $\varepsilon = \varepsilon_{n+1}$, and $\beta = \beta_{n+1}$. So we get a partitioning $\mathfrak{U}_\xi^n = \{U_{\eta}^n : 0 \leq \eta < \eta_\xi^n\}$ of U_ξ^n to every element of which a nonnegative real, an equivalent norm, and a $p_{U_\xi^n}$ -norm-one vector are associated.

Now $\mathfrak{U}^{n+1} = \bigcup\{\mathfrak{U}_\xi^n : 0 \leq \xi < \xi_n\}$ is a relatively open partitioning of B^* if its elements $U_{\xi\eta}^n$, $0 \leq \xi < \xi_n$, $0 \leq \eta < \eta_\xi^n$, are ordered lexicographically (see [5, Proposition 1.5]). Obviously \mathfrak{U}^{n+1} is a refinement of \mathfrak{U}^n .

Step 4. Proof that \mathfrak{U} is separating. Let us assume the contrary, i.e., there exist two different points x_1^* and x_2^* in B^* such that they belong to one and the same element V_n of \mathfrak{U}^n for every positive integer n . Let us denote by s_n , p_n , and e_n respectively the nonnegative real number, the equivalent norm, and the unit vector associated to V_n by the construction.

Our assumption implies that neither V_n is empty nor $V_n = \{0\}$ and hence $s_n > 0$ for every $n \in \mathbb{N}$. The construction gives us that the following properties hold:

- (i) $B^* = V_0 \supset V_1 \supset V_2 \supset \dots \supset \bigcap_{n=1}^\infty V_n \supset \{x_1^*, x_2^*\}$;
- (ii) $V_n \subset \{x^* \in V_{n-1} : \langle x^*, e_n \rangle > (1 - \varepsilon_n)s_n\}$ for every $n \geq 1$;
- (iii) $s_n = \sup\{p_{n-1}^*(x^*) : x^* \in V_n\} > 0$;
- (iv) $p_{n-1}(e_n) = 1$ for every $n \geq 1$;
- (v) $p_0 = \|\cdot\|$, $p_n^2 = p_{n-1}^2 + \beta_n \cdot q_n^2$, where $q_n(x) = \inf\{\|x - \lambda e_n\| : \lambda \in \mathbb{R}\}$ for every $x \in E$.

Further the proof is contained in [PPhN] (the last half of the proof of Theorem 1.6) with the same notation except for the indexation of the smooth norms which is moved back by one.

1.2. Remark. Theorem 2.2 of [PPhN] states that the dual (E^*, w^*) of a smooth space E belongs to the Stegall's class S , i.e., every minimal usco correspondence from a Baire space into it is single-valued on a residual subset of its

domain. (See the definition of “minimal usco” below. For additional information about S see for example [S, S1, K, ČK].) Lemma 5.10 from [N] (or [R, Proposition 2.5]) shows that fragmentable spaces are in S and so Theorem 1.1 immediately yields Theorem 2.2 of [PPhN].

2. β -DIFFERENTIABILITY AND β -SMALLNESS

2.1. Definition. Let X and Y be Hausdorff topological spaces and $F: X \rightarrow Y$ be a multivalued map. F is said to be upper semicontinuous at $x_0 \in X$ if for every open set $V \subset Y$ that contains $F(x_0)$ there exists a neighborhood U of x_0 such that $F(x) \subset V$ for every $x \in U$. We say that F is upper semicontinuous if it is upper semicontinuous at every point $x \in X$. If, in addition, $F(x)$ is nonempty and compact for every x in X , the correspondence F is called usco.

The graphs of all usco correspondences from X into Y can be ordered by inclusion. Minimal elements of this relation are graphs of usco correspondences from X into Y , which will be called “minimal usco”. For such mappings the following property holds: for every open subset U of X and every open subset V of Y such that $F(U) \cap V \neq \emptyset$ there exists a nonempty open subset W of U with $F(W) \subset V$ (see [ChK]).

2.2. Definition. Let β be a family of nonempty bounded subsets S of the Banach space E satisfying:

- (a) $\lambda S \in \beta$ whenever $\lambda \in \mathbb{R}$ and $S \in \beta$;
- (b) the union of finitely many members of β is in β ;
- (c) the union of the members of β is all of E .

We say that a continuous convex function f on an open convex subset D of E is β -differentiable at $x \in D$ ($(f(x+ty) - f(x))/t$) if for all $S \in \beta$ the limit of the difference quotient $\lim_{t \rightarrow 0} (f(x+ty) - f(x))/t$ exists uniformly for y in S . Natural choices for β are all finite sets (Gâteaux differentiability), all weakly compact sets (Hadamard differentiability), or all bounded sets (Fréchet differentiability). If the norm is β -differentiable at every $x \neq 0$, it is said to be β -smooth.

We say that $U \subset E^*$ is a neighborhood of $x^* \in E^*$ if there exists $S \in \beta$ with $x^* + S^0 \subset U$, where $S^0 = \{z^* \in E^*: \langle z^*, z \rangle \leq 1 \text{ for all } z \in S\}$. It is easy to check that thus we have introduced a linear topology in E^* . We will call it β as well. This topology is closely related to β -differentiability by the following fact: The subdifferential mapping ∂f of a continuous convex function $f: D \rightarrow \mathbb{R}$ (D is an open subset of E) is single-valued and β -upper semicontinuous at the point $x_0 \in D$ if and only if f is β -differentiable at x_0 (see [PPhN, Proposition 3.1]).

We will need the following

2.3. Definition. Let X be a topological space and $\mathfrak{U} = \bigcup_{n=1}^{\infty} \mathfrak{U}^n$ be a σ -relatively open partitioning of X . Let τ be a second topology on X . We say that \mathfrak{U} is τ -small if for every $x \in X$ and for every neighborhood W of x there exist a positive integer n and an element U of \mathfrak{U}^n with $x \in U \subset W$.

2.4. Remark. Let ρ be a metric on the topological space X and τ_ρ be the topology generated by ρ . If ρ fragments X then there exists a separating

τ_ρ -small σ -relatively open partitioning of X . On the other hand, if X admits a separating τ_ρ -small σ -relatively open partitioning, then ρ fragments all compact subsets of X .

The following proposition justifies involving the notion of τ -smallness. It is a generalization of [R, Proposition 2.5].

2.5. Proposition. *Let (X, τ_1) be a topological space, τ_2 be a second topology on X and suppose X admits a separating τ_2 -small τ_1 - σ -relatively open partitioning. Then every minimal usco map $\varphi: B \rightarrow (X, \tau_1)$ from a Baire space B into (X, τ_1) is single-valued and τ_2 -upper semicontinuous at every point of a residual subset of B .*

Proof. Let $\mathfrak{U} = \bigcup_{n=1}^\infty \mathfrak{U}^n$, $\mathfrak{U}^n = \{U_\xi^n : 0 \leq \xi < \xi^n\}$ be a separating τ_2 -small τ_1 - σ -relatively open partitioning of X . We put $B_n = \{b \in B : \text{there exist a neighborhood } V \text{ of } b \text{ and an element } U \text{ of } \mathfrak{U}^n \text{ such that } \varphi(V) \subset U\}$ for every $n \in \mathbb{N}$. Obviously the B_n are open subsets of B . We will prove that they are dense. Indeed, let V be an arbitrary nonempty open subset of B and n be a positive integer. We consider the ordinal number

$$\hat{\xi} = \min\{\xi : U_\xi^n \cap \varphi(V) \neq \emptyset\}.$$

The minimality of φ and $\varphi(V) \cap (\bigcup_{\eta < \hat{\xi} + 1} U_\eta^n) \neq \emptyset$, $\bigcup_{\eta < \hat{\xi} + 1} U_\eta^n$ open yield the existence of a nonempty open subset W of V with $\varphi(W) \subset \bigcup_{\eta < \hat{\xi} + 1} U_\eta^n$. Hence $\varphi(W) \subset U_{\hat{\xi}}^n$ because of the choice of $\hat{\xi}$ and so $W \subset B_n$, i.e., $B_n \cap V \neq \emptyset$.

Now we will prove that φ is single-valued and τ_2 -continuous at every point b in the residual subset $G = \bigcap_{n=1}^\infty B_n$ of B . As $\varphi(b)$ is contained in one element of \mathfrak{U}^n for every n , no two points from $\varphi(b)$ can be separated by \mathfrak{U} . Therefore $\varphi(b)$ is a singleton (\mathfrak{U} is separating). Let U be a τ_2 -open neighborhood of $\varphi(b)$. Then there exist n and $\alpha_n < \xi_n$ with $\varphi(b) \in U_{\alpha_n}^n \subset U$ because \mathfrak{U} is τ_2 -small. But $b \in G \subset B_n$, and hence there exists a neighborhood V of b with $\varphi(V) \subset U_{\alpha_n}^n$. So $\varphi(V) \subset U$ and φ is τ_2 -upper semicontinuous at b .

2.6. Corollary. *Let E be a Banach space and β be a family of its subsets as described in the beginning of this section. If (E^*, w^*) admits a separating β -small σ -relatively open partitioning, then every continuous convex function on an open subset of E is β -differentiable on a residual subset of its domain.*

One can prove this using [ChK, Proposition 2.5, Lemma 2.5] or [PPhN, Lemma 3.5].

2.7. Theorem. *Suppose that the norm in E is β -smooth. Then (E^*, w^*) admits a separating β -small σ -relatively open partitioning.*

Because of Proposition 2.5, this theorem is a generalization of [PPhN, Theorem 3.4] but its proof is the same. One should skip the beginning of the proof in [PPhN] and use the subsequent argument to show that the σ -relatively open partitioning constructed in the proof of Theorem 1.1 is actually β -small.

Combining the results from this section one can obtain once more the following

2.8. Corollary. *Let E be a Banach space whose norm is Fréchet smooth. Then the dual norm fragments the bounded subsets of (E^*, w^*) .*

ACKNOWLEDGMENT

The author is very grateful to Professor P. S. Kenderov for his fruitful discussions and encouragement.

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