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ON A CONJECTURE OF NITSCHE

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ABSTRACT. We show that under the hypothesis of bounded Gaussian curvature, certain minimal surfaces are in fact of finite total curvature. We can then answer the following version of a conjecture of Nitsche (J. Math. Mech. 11 (1962), 295) under the hypothesis of bounded Gaussian curvature:

Conjecture. *Let $M^2 \subset \mathbf{R}^3$ be a complete minimal surface such that for some height function H, the level sets are (compact)* <u>*Jordan curves*</u>*. Then M is a catenoid.*

Much is known about minimal surfaces that are of finite total curvature [JM, O, S]. We show that under the hypothesis of bounded Gaussian curvature, certain minimal surfaces are in fact of finite total curvature. With this information we can then answer the following version of a conjecture of Nitsche [Ni], under the hypothesis of bounded Gaussian curvature:

Conjecture. *Let $M^2 \subset \mathbf{R}^3$ be a complete minimal surface such that for some height function H, the level sets are (compact)* <u>*Jordan curves*</u>*. Then M is a catenoid.*

In [CHM, p. 1], an example is given, due to Riemann, whose level sets are only round circles and straight lines. This example shows that the conjecture would be false without the compactness hypothesis. Let S^2 denote the Riemann sphere. The conjecture (with $|K| < C$) follows from the following deeper therom

Theorem 1. *Let $M^2 \subset \mathbf{R}^3$ be a complete minimal immersion satisfying*

(i) $|K| < C$ *(M is of bounded Gaussian curvature).*

(ii) *The immersion is given by $I = (I_1, I_2, I_3)$: $S^2 - \{p, q\} \to \mathbf{R}^3$ and is such that the limits as $z \to p$ and as $z \to q$ of $I_3(z)$ exist uniformly as extended real numbers.*

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Then M is of finite total curvature. In particular, if M is embedded, then M is a catenoid.

1. Introduction

We begin with the Weierstrass representation of minimal surfaces [O]

Theorem. *Let $\Delta \subseteq \mathbf{C}$ be an open connected subset. Let $f\colon \Delta \to \mathbf{C}$ be holomorphic and $g\colon \Delta \to \mathbf{C} \cup \{\infty\}$ be meromorphic. Suppose that at each pole of g of order m there is a zero of f of order $2m$ and that $f(1-g^2)$, $if(1+g^2)$, and fg have no real periods. Then there is a minimal immersion $I = (I_1, I_2, I_3)\colon \Delta \to \mathbf{R}^3$ given by*

$$I_1 = \operatorname{Re}\frac{1}{2}\int_*^z f(1-g^2), \qquad I_2 = \operatorname{Re}\frac{i}{2}\int_*^z f(1+g^2), \qquad I_3 = \operatorname{Re}\int_*^z fg.$$

Conversely, every minimal surface is of this form.

The metric on this surface is given by $ds^2 = \frac{1}{2}|f|^2(1+|g|^2)^2|dz|^2$. The function g given in the representation is (up to stereographic projection) the Gauss map of the surface. A minimal surface is said to be complete if every curve converging to the boundary of Δ is of infinite length. The Gaussian curvature of a minimal surface given by a Weierstrass representation is

$$K = -\left(\frac{4|g'|}{|f|\,(1+|g|^2)^2}\right)^2.$$

We will denote $\mathbf{C} - \{0\}$ by $\mathbf{C}^*$. It will be useful in our proofs to know what happens when we take $fg = 1/z$ in the Weierstrass representation

Proposition 1. *Suppose that in the Weierstrass representation we have a domain $\Delta = \mathbf{C}^*$, $f\colon\Delta \to \mathbf{C}$ holomorphic, and $g\colon\Delta \to \mathbf{C} \cup \{\infty\}$ meromorphic. If we require that $fg(z) = 1/z$ then neither f nor g have any poles or zeros in $\mathbf{C}^*$.*

Proof. Immediate.

Under certain circumstances (stated in Lemma 3), we can restrict our attention to minimal surfaces generated by a small class of $\{f, g\}$'s and to the single domain $\mathbf{C}^*$. We begin with two preliminary lemmas.

Lemma 1. *Let Δ be a Riemann surface that is homeomorphic to $S^2 - \{p, q\}$ with $U\colon \Delta \to \mathbf{R}$ harmonic. Suppose $\lim_{z\to p} U(z) = -\infty$ and $\lim_{z\to q} U(z) = \infty$. Then Δ is conformally equivalent to $\mathbf{C}^*$.*

Proof. Observe that $\pi_1(\Delta) = \mathbf{Z}$. Let $\widetilde{\Delta}$ be the (holomorphic) universal cover of Δ. Then $\widetilde{\Delta}$ is conformally equivalent to either $\mathbf{C}$ or $D = \{|z| < 1\}$. In the first case, since $\pi_1(\Delta) = \mathbf{Z}$, it follows that Δ is biholomorphic to $\mathbf{C}$ modulo a translation group; that is, Δ is conformally $\mathbf{C}^*$.

We proceed to show that $\widetilde{\Delta}$ could not be D. Again, since $\pi_1(\Delta) = \mathbf{Z}$, Δ is conformally D modulo $[g]$, where g is a complex automorphism of D without fixed points. It follows that, in the classical terminology, g is hyperbolic if g has two fixed points on ∂D or parabolic if it has one fixed point. In the hyperbolic case, the fundamental region for the group $[g]$ is the complement of two disks in D tangential to the fixed points on ∂D. In the parabolic case, the fundamental region lies between two geodesics emanating from the fixed point. Label the fundamental region (in both cases) R. Let ρ: $D \to \Delta$ be the natural projection, and note that $\widetilde{U} = U \circ \rho$: $R \to \mathbf{R}$ is harmonic and tends to $\pm\infty$ on an arc of ∂D. Let $\widetilde{V}$ denote the harmonic conjugate of $\widetilde{U}$ on D. Now either $f_1 = e^{(\widetilde{U}+i\widetilde{V})}$ or $f_2 = e^{-(\widetilde{U}+i\widetilde{V})}$ tends to zero on this arc. By the reflection principle this holomorphic function is identically zero on D. This is a contradiction. Q.E.D.

Lemma 2. *Let* U: $\mathbf{C}^* \to \mathbf{R}$ *be harmonic, and suppose that* $\lim_{|z|\to 0} U(z) = -\infty$ *and* $\lim_{|z|\to\infty} U(z) = \infty$. *Then* $U(re^{i\theta}) = a\log(r) + b$ *where* $a, b \in \mathbf{R}$.

Proof. By elementary complex analysis.

Lemma 3. *Let* M *be a complete minimal surface satisfying condition* (ii) *of Theorem* 1 *and such that* I_3: $S^2 - \{p, q\} \to \mathbf{R}$ *is surjective. Then, up to translation, homothety, and relabeling of* p *and* q, *we have that* M *is given by a Weierstrass representation where:* (1) $\Delta = \mathbf{C}^*$ *and* (2) $fg(z) = 1/z$.

Proof. Clearly we have $\lim_{z\to p} I_3(z) = -\infty$ and $\lim_{z\to q} I_3(z) = \infty$, up to relabeling of p and q. By Lemma 1, $S^2 - \{p, q\}$ is conformally equivalent to $\mathbf{C}^*$. Thus in the Weierstrass representation we may take $\Delta = \mathbf{C}^*$. So we might as well have started with M being a minimal immersion of $\mathbf{C}^*$ with I_3: $\mathbf{C}^* \to \mathbf{R}$ satisfying $\lim_{|z|\to 0} I_3(z) = -\infty$ and $\lim_{|z|\to\infty} I_3(z) = \infty$. By Lemma 2, $I_3(re^{i\theta}) = a\log(r) + b$ for some $a, b \in \mathbf{R}$.

Thus M can be given in the Weierstrass representation by $\widetilde{I} = (\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3)$: $\mathbf{C}^* \to \mathbf{R}^3$, where we have taken $\widetilde{I}_3(z) = I_3(z)$. That is, $I_3(z) = a\log(r) + b = \widetilde{I}_3(z) = \operatorname{Re}\int_*^z fg\,dz$. After translation, we may discard b. We are left with $\widetilde{I}_3(z) = a\log(r)$. Applying a homothety (by $1/a$) yields

$$\frac{1}{a}\widetilde{I}_3 = \frac{1}{a}\operatorname{Re}\int_*^z fg\,dz = \log(r).$$

We can assume $a = 1$, and therefore take $fg = 1/z$. Q.E.D.

We will also use a result of Lehto and Virtanen on the growth aspects of meromorphic functions. Let h: $\Delta \subseteq \mathbf{C} \to \mathbf{C} \cup \{\infty\}$ be a meromorphic function, where Δ is open and connected. The spherical derivative of h is defined as

$$\rho(h(w)) = \frac{|h'(w)|}{\left[1 + |h(w)|^2\right]}.$$

Lemma 4. *Let* h *be a meromorphic function in a neighborhood of an essential singularity at infinity that satisfies the inequality:* $\limsup_{z\to\infty} |z|\,\rho(h(z)) < \infty$. *Then* h *cannot omit any value.*

Proof [LV2, p. 7,8]. Let γ be a Jordan path in U tending to ∞. Then α is said to be an asymptotic value of h at ∞ if $h(z) \to \alpha$ as $z \to \infty$ along γ. Suppose h omits the value α. Then by Iversen's Theorem [No, p. 4], α is an asymptotic value at infinity along a Jordan path γ. By the above theorem, h is normal in U slit along the path γ. By Theorem 2 of [LV1, p. 53] and the remark that follows, h converges uniformly in $U - \{\gamma\}$ toward α, no matter how z goes to ∞. This contradicts the hypothesis that h has an essential singularity at $z = \infty$. Hence h cannot omit any value. Q.E.D.

If h omits the value 0 and satisfies the inequality, then it could not have had an essential singularity at $z = \infty$.

2. The result

The following theorems are needed for the proof of Theorem 1.

Theorem (Jorge-Meeks [JM]). *The catenoid is the only embedded complete minimal annulus in* $\mathbf{R}^3$ *that has finite total curvature.*

Theorem (Xavier [X]). *Let* $M^2 \subset \mathbf{R}^3$ *be a complete minimal surface that is nonflat and of bounded Gaussian curvature* ($|K| < C$). *Then the convex hull of* M *is* $\mathbf{R}^3$.

Corollary. *Let* M *be a complete minimal immersion satisfying conditions* (i) *and* (ii) *of Theorem* 1. *Then* I_3 *is surjective.*

Proof (*of Theorem* 1). By the corollary to Xavier's Theorem, I_3 is surjective. By Lemma 3, we have that, up to translation, homothety, and the relabeling of p and q, M is given in the Weierstrass representation by $\Delta = \mathbf{C}^*$ and $fg(z) = 1/z$. By Proposition 1, f has no zeros in $\mathbf{C}^*$. Thus we can set $(\frac{1}{|f|}) = |zg|$ and $z = re^{i\theta}$. Setting $h(z) = g(z)^2$, the above expression for Gaussian curvature becomes

$$K = -\left(\frac{2r\,|h'|}{(1+|h|)^2}\right)^2.$$

Thus,

$$|K| < C \Rightarrow \left(\frac{r\,|h'|}{(1+|h|)^2}\right) < C \iff \left(\frac{|h'|}{(1+|h|)^2}\right) < \frac{C}{r}$$
$$\iff \left(\frac{|h'|}{1+|h|^2}\right) < \frac{C_1}{r}.$$

In terms of the spherical derivative, we have

$$|K| < C \Rightarrow |z|\,\rho(h(z)) < C_1 \quad \forall\, z \in \mathbf{C}^*.$$

By Proposition 1, g (and hence h) has no zeros or poles in $\mathbf{C}^*$. All that remains is to check for singularities at 0 and ∞. Suppose that h has an essential singularity at ∞. We have just shown that

$$|z|\,\rho(h(z)) < C \quad \forall\, z \in \mathbf{C}^*, \qquad \text{so } \limsup_{|z|\to\infty} |z|\,\rho(h(z)) < C < \infty.$$

Thus by Lemma 4, h cannot omit any value. This is a contradiction since we have shown that h omits zero. Hence h could not have had an essential singularity at infinity. Suppose now that h has an essential singularity at $z = 0$.

Then by the inversion $z \mapsto \frac{1}{z}$ we have that $h(\frac{1}{z})$ has an essential singularity at ∞. Observe that

$$\left(\frac{|z||h'(z)|}{1+|h(z)|^2}\right) < C \quad \forall z \in \mathbf{C}^* \iff \left(\frac{|z|\left|(h(\frac{1}{z}))'\right|}{1+|h(\frac{1}{z})|^2}\right) < C \quad \forall z \in \mathbf{C}^*$$

By the above argument, h could not have had an essential singularity at zero.

We have that h has a pole or removable singularity at $z=0$ and $z=\infty$. Hence h is meromorphic on $\mathbf{C} \cup \{\infty\}$ and therefore rational [F, p. 11]. This implies that g is also rational. The image of $\mathbf{C}^*$ under g has finite area on S^2; that is, we have a complete minimal surface M of finite total curvature.

If M is embedded, apply the result of Jorge-Meeks [JM] to conclude that M is a catenoid. Q.E.D.

We have, as a corollary to Theorem 1, that Nitsche's conjecture holds under the hypothesis of bounded Gaussian curvature.

Theorem 2. *Let M be a complete minimal immersion satisfying conditions* (i) *and* (ii) *of Theorem* 1. *Suppose that the level sets of I_3: $S^2 - \{p, q\} \to \mathbf{R}$ are homeomorphic to S^1. Then M is a catenoid.*

Proof. M is embedded, so by Theorem 1, M is a catenoid. Q.E.D.

Rosenberg has given a result on the unwillingness of the catenoid to be perturbed [R]. Certain definitions are needed and are given as follows. Let T_ε be a tubular neighborhood of radius $\varepsilon > 0$ about $M \subset \mathbf{R}^3$. $\widetilde{M}$ is said to be an ε-C^1 variation of M if $\widetilde{M}$ is a graph over M which is in T_ε and is C^1 close to M. This means $\widetilde{M}$ is pointwise ε-close to M in each fibre of T_ε and in the tangent planes as well. M is said to be isolated if for some $\varepsilon > 0$, the only ε-C^1 variations of M differ from M by an isometry of $\mathbf{R}^3$. With these definitions, he showed that the catenoid is ε-C^1 isolated.

Theorem 1 is of the same flavor as this result and leads toward the conjecture that the catenoid is not only ε-C^1 isolated but is in fact ε-C^0 isolated.

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