A GEOMETRIC INTERPRETATION OF SEGAL'S INEQUALITY $\|e^{X+Y}\| \leq \|e^{X/2}e^Ye^{X/2}\|

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Abstract. It is shown that the exponential mapping of the manifold of positive elements of a $C^*$-algebra (provided with its natural connection) increases distances (when measured in the natural Finsler structure). The proof relies on Segal's inequality $\|e^{X+Y}\| \leq \|e^{X/2}e^Ye^{X/2}\|$, valid for all symmetric $X, Y$ in any $C^*$-algebra. In turn, this geometric inequality implies Segal's inequality.

Let $A$ be a $C^*$-algebra with $1$ and denote by $A^+$ the set of positive invertible elements of $A$. Then $A^+$ is an open subset of $A^+$, the real Banach space of symmetric elements in $A$, and therefore, the tangent space $T_aA^+$ to the manifold $A^+$ at $a \in A^+$ can be identified to $A^+$. The manifold $A^+$ carries a natural Finsler metric (see [1]) defined by $\|X\|_a = \|a^{-1/2}Xa^{-1/2}\|$ for $X \in TA^+_a$. This norm corresponds to the following interpretation: assume $A$ is faithfully represented in a Hilbert space $(L, (\ , \ ))$, and for each $a \in A^+$, define an inner product in $L$ by $(\xi, \eta)_a = (a^{-1/2}\xi, \eta)$. On the other hand, each $X \in TA^+_a$ determines the bilinear form $B(\xi, \eta) = (X\xi, \eta)$ on $L$. Then the Finsler norm $\|X\|_a$ coincides with the norm of the bilinear form $B$ in the Hilbert space $(L, (\ , \ ))$.

The group $G$ of invertible elements of $A$ acts on $A^+$ by $\mathcal{M}_g a = (g^*)^{-1}ag^{-1}$, $(g \in G, a \in A^+)$ making $A^+$ into a reductive homogeneous space (see [2]) with the natural connection given by

$$D_X Y = X(Y) - \frac{1}{2}(Xa^{-1}Y + Ya^{-1}X),$$

where $X(Y)$ denotes the derivative of the field $Y$ in the direction $X$ in the Banach space $A^+$. In this connection, the geodesic $\gamma$ with $\gamma(0) = a$ and $\dot{\gamma}(0) = X$ has the form $\gamma(t) = e^{tXa^{-1/2}}ae^{tXa^{-1/2}}$.

Further, for each $g \in G$ and $a \in A^+$, the map $g$ is an isometry from the Hilbert space $(L, (\ , \ ))$ onto $(L, (\ , \ ))_{g,a}$ and consequently the tangent map $(T\mathcal{M}_g)_a : TA^+_a \to TA^+_{g,a}$ is an isometry for the Finsler metric.

The geometry of $A^+$ in this setting was studied in [1] where, in particular, the following result is proved [1, Theorem 6.3]: the distance $d(a, b)$ in the Finsler metric defined by $d(a, b) = \inf\{\text{length}(\gamma) ; \gamma$ joins $a$ to $b\}$,
is given by \( d(a, b) = \text{length of the unique geodesic in } A^+ \text{ joining } a \text{ to } b \), i.e.,

\[ d(a, b) = \|X\|_a \text{ where } b = e^{Xa^{-1/2}ae^{-a^{-1}x/2}}. \]

Notice that the Finsler structure in \( A^+ \) does not come from a Riemannian metric. However, \( A^+ \) shares with Riemannian manifolds of nonpositive curvature the following metric property, which is the main result of this note.

**Theorem 1.** For each \( a \in A^+ \), the exponential map \( \exp_a : TA_a \to A^+ \) increases distances in the sense that

\[ d(\exp_a X, \exp_a Y) \geq \|X - Y\|_a \]

for all \( X, Y \in TA_a \).

**Proof.** Since \( G \) acts isometrically, it suffices to prove the inequality for \( a = 1 \).

Set \( x = \exp_1 X = e^X \), \( y = \exp_1 Y = e^Y \). The geodesic that joins \( x \) to \( y \) in time 1 has the formula

\[ y(t) = e^{Zx^{-1/2}x^{-1/2}Zt/2}, \]

where \( b = y(1) = e^{Zx^{-1/2}x^{-1/2}Zt/2} \). The inequality we are after is

\[ \|X - Y\| \leq \|Z\|_x = \|x^{-1/2}Zx^{-1/2}\| \]

or

\[ \|\log x - \log y\| \leq \|x^{-1/2}Zx^{-1/2}\|. \]

But

\[ x^{-1/2}yx^{-1/2} = x^{-1/2}(e^{Zx^{-1/2}x^{-1/2}Z/2})x^{-1/2} = e^{(x^{-1/2}Zx^{-1/2})/2}e^{(x^{-1/2}Zx^{-1/2})/2}x^{-1/2} = e^{x^{-1/2}Zx^{-1/2}}. \]

Then \( x^{-1/2}Zx^{-1/2} = \log(x^{-1/2}yx^{-1/2}) \) so we must prove \( \|\log x - \log y\| \leq \|\log(x^{-1/2}yx^{-1/2})\| \) or, changing \( x \) into \( x^{-1} \),

\[ \|\log x + \log y\| \leq \|\log(x^{-1/2}yx^{-1/2})\|. \]

Replacing \( x, y \) by \( kx, ky \) with \( k \) a large positive number allows us to assume without loss of generality that \( \log x \geq 0 \) and \( \log y \geq 0 \). Then, the last inequality is equivalent to

\[ \|e^{\log x + \log y}\| \leq \|x^{1/2}yx^{1/2}\|. \]

But this is equivalent to Segal's inequality (see [3, Theorem X.57, p. 260, vol. II], or [4])

\[ \|e^{X+Y}\| \leq \|e^{X/2}e^Ye^{X/2}\| \]

and this concludes the proof of Theorem 1. Obviously all steps in the proof can be reversed, so that \((**) \) implies \((*) \).

As an application of Theorem 1, consider a \( C^* \)-algebra \( A \) with a distinguished family \( p_1, p_2, \ldots, p_n \) of selfadjoint orthogonal projections satisfying \( p_ip_j = 0 \) if \( i \neq j \) and \( p_1 + p_2 + \cdots + p_n = 1 \). Let \( B \subset A \) be the \( C^* \)-subalgebra of elements of \( A \) that commute with all \( p_i \) and \( H \subset A \) be the Banach subspace of elements \( h \in A \) satisfying \( p_ihp_i = 0 \) for each \( i \). Let also \( E = \{e^h : h = h^* \in H\} \).
Theorem 2. For each $b > 0$ in $B$, the distance (in the Finsler metric) from $b$ to the submanifold $E \subset A^+$ is attained at $1 \in E$.

Proof. Set $X = \log b$. By definition $X = X_1 + \cdots + X_n$, where $X_i = p_i X_p_i$. Since $\|X\| = \max \|X_i\|$, we can assume that $\|X\| = \|X_1\|$, and accordingly, we choose a faithful representation of $A$ in a Hilbert space $L$ with the additional property that, setting $L = L_1 \oplus \cdots \oplus L_n$ with $L_i = p_i(L)$, the subspace $L_1$ contains a "norming eigenvector" for $X_1$, i.e., a unit vector $\xi \in L_1$ with $X_1 \xi = \pm \|X_1\| \xi$. Let $Y$ be an arbitrary selfadjoint element of $H$. Then, by the definition of $H$, $Y \xi \in L_2 \oplus \cdots \oplus L_n$ and therefore $X \xi = X_1 \xi$ is perpendicular to $Y \xi$. As a consequence we have

$$d(b, 1) = \|X\| = \|X \xi\| \leq \|X_1 \xi - Y \xi\| \leq \|X - Y\|.$$ 

Then using Theorem 1, we conclude that $d(b, 1) \leq d(b, e^Y)$ and we are done.

Remark. Notice that the tangent map to $\exp$ also increases norms. In fact it suffices to show this property for $a = 1$. For that we estimate

$$\left\| \frac{e^{Y+tZ} - e^Y}{t} \right\|_{e^Y} = \frac{1}{t} \| e^{-Y/2} e^{Y+tZ} e^{-Y/2} - I \|$$

using Segal's inequality $\| e^{-Y/2} e^{Y+tZ} e^{-Y/2} - e^{t \|Z\|} \| \geq \| e^{t \|Z\|} \|$. Assume that $t > 0$ and that $\max \text{Spec}(Z) = \|Z\|$. Then $\| e^{t \|Z\|} \| = e^{t \|Z\|} \geq 1$. Hence in this case

$$\frac{1}{t} \| e^{-Y/2} e^{Y+tZ} e^{-Y/2} - I \| = \frac{1}{t} (\| e^{t \|Z\|} - 1 \|) \geq \frac{1}{t} (\| e^{t \|Z\|} - 1 \|) \geq \|Z\|.$$

Then

$$\lim_{t \to 0^+} \left\| \frac{e^{Y+tZ} - e^Y}{t} \right\|_{e^Y} \geq \|Z\|.$$ 

For other $Z$'s, change $Z$ into $-Z$.

References


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