THE EXISTENCE OF LEFT AVERAGING FUNCTIONS THAT ARE NOT RIGHT AVERAGING

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Abstract. Let $G$ be a locally compact group. We show that $G$ is amenable as a discrete group if and only if $\sum_{i=1}^{n} \lambda_{i} x_{i} f \in A_{0}$ for any $f_{0} \in A_{0}$, $x_{i} \in G$, and $\lambda_{i} > 0$ ($i = 1, 2, \ldots, n$) with $\sum_{i=1}^{n} \lambda_{i} = 1$, where $A_{0}$ is the set of functions that left average to 0. We also confirm a conjecture of Rosenblatt and Yang that there is a left averaging function that is not right averaging if $G$ is not amenable.

Let $G$ be a locally compact group with a fixed left Haar measure $\lambda$. Let $L^{p}(G)$ be the associated real Lebesgue spaces ($1 \leq p \leq \infty$). For each $f \in L^{\infty}(G)$ and $x \in G$, $xf \in L^{\infty}(G)$ is defined by $xf(y) = f(xy)$, $y \in G$. Similarly, we can define $f_{x} \in L^{\infty}(G)$ by $f_{x}(y) = f(yx)$, $y \in G$. For $f \in L^{\infty}(G)$ and a constant $c$, we say that $f$ left averages to $c$ if $c \in \| \cdot \|_{\infty}$-closed convex hull of $\{xf : x \in G\}$. Let $\mathcal{A}$ denote the set of all functions that left average to some constant, and let $A_{0}$ denote the set of functions that left average to 0. Similarly, we say that $f$ right averages to $c$ if $c \in \| \cdot \|_{\infty}$-closed convex hull of $\{f_{x} : x \in G\}$, and we denote the set of all the right averaging functions by $\mathcal{R}$. A positive linear functional of norm 1 on $L^{\infty}(G)$ is called a mean. A mean $m$ is said to be left invariant if $m(xf) = m(f)$ for any $x \in G$ and $f \in L^{\infty}(G)$. We denote the set of all left invariant means by $\text{LIM}(G)$. $G$ is said to be amenable if $\text{LIM}(G) \neq \emptyset$. It is well known that $G$ is amenable if $G$ is amenable as a discrete group, but the converse may fail (see [1]).

It is natural to ask if every left averaging function is right averaging. This problem was first studied by Rosenblatt and Yang [3]. They showed that when $G$ is amenable as a discrete group, $\mathcal{A} = \mathcal{R}$ if and only if $\text{LIM}(G) = \text{RIM}(G)$, where $\text{RIM}(G)$ is the set of all right invariant means i.e. the mean $m$ with $m(f_{x}) = m(f)$ for any $x \in G$ and $f \in L^{\infty}(G)$. They conjectured that when $G$ is not amenable, there is a left averaging function that is not right averaging.

In this note we prove that $G$ is amenable as a discrete group if and only if $\sum_{i=1}^{n} \lambda_{i} x_{i} f \in A_{0}$ for any $f \in \mathcal{A}$, $x_{i} \in G$, and $\lambda_{i} > 0$ with $\sum_{i=1}^{n} \lambda_{i} = 1$.
This improves the main result of Miao [2]. Then we apply this theorem to confirm the conjecture of Rosenblatt and Yang. To prove our result, we need the following lemmas.

**Lemma A.** (a) For $f \in L^\infty(G)$, let $\tilde{f} \in L^\infty(G)$ be defined by $\tilde{f}(x) = f(x^{-1})$ ($x \in G$). Then $f \in \mathcal{A}$ if and only if $\tilde{f} \in \mathcal{A}_R$.

(b) $\mathcal{A} \subseteq \mathcal{A}_R$ if and only if $\mathcal{A} = \mathcal{A}_R$.

*Proof.* (a) For any $x \in G$ and $t \in G$,

$$(x \tilde{f})(t) = f(tx^{-1}) = \tilde{f}(tx^{-1}) = \tilde{f}_{x^{-1}}(t).$$

Hence $(x \tilde{f}) = \tilde{f}_{x^{-1}}$. It is easy to see that the following are equivalent: (i) $f \in \mathcal{A}$; (ii) there is a constant $c$ such that $c \in \text{convex}^\mathbb{R}\{x f : x \in G\}$; (iii) $c \in \text{convex}^\mathbb{R}\{(x \tilde{f}) : x \in G\} = \text{convex}^\mathbb{R}\{\tilde{f}_{x^{-1}} : x \in G\}$; and (iv) $\tilde{f} \in \mathcal{A}_R$.

(b) Let $\mathcal{A} \subseteq \mathcal{A}_R$. If $f \in \mathcal{A}_R$ then $\tilde{f} \in \mathcal{A}$ by (a) since $(\tilde{f}) = f$. Hence $f \in \mathcal{A} \subseteq \mathcal{A}_R$ and $\tilde{f} \in \mathcal{A}_R$ by (a) again. Therefore $\mathcal{A} = \mathcal{A}_R$. $\square$

**Lemma B.** $\mathcal{A}_0$ is a subspace if and only if for any $f \in \mathcal{A}_0$, $\sum_{i=1}^n \lambda_i x_i f \in \mathcal{A}_0$ for any $x_i \in G$, $\lambda_i > 0$ ($i = 1, 2, \ldots, n$) with $\sum_{i=1}^n \lambda_i = 1$.

*Proof.* Suppose that $\mathcal{A}_0$ is a subspace. If $f \in \mathcal{A}_0$, then $x f \in \mathcal{A}_0$ for any $x \in G$. Let $x_i \in G$ and $\lambda_i > 0$ ($i = 1, 2, \ldots, n$), then $\lambda_i x_i f \in \mathcal{A}_0$ ($i = 1, 2, \ldots, n$). So $\sum_{i=1}^n \lambda_i x_i f \in \mathcal{A}_0$ because $\mathcal{A}_0$ is a subspace.

Conversely, let $\sum_{i=1}^n \lambda_i x_i f \in \mathcal{A}_0$ for any $f \in \mathcal{A}_0$, $x_i \in G$, and $\lambda_i > 0$ ($i = 1, 2, \ldots, n$) with $\sum_{i=1}^n \lambda_i = 1$. It suffices to show that $\mathcal{A}_0 \supseteq \text{span}\{x f - f : f \in L^\infty(G), x \in G\}$ by Lemma 2.1 in [2]. For $f_i \in L^\infty(G)$ and $x_i \in G$, $x_i f - f \in \mathcal{A}_0$. Let $F = \sum_{i=1}^{n-1} a_i (x_i f_i - f_i) \in \mathcal{A}_0$ for any constants $a_i$, $x_i \in G$ and $f_i \in L^\infty(G)$ ($i = 1, 2, \ldots, n-1$). If $x_n \in G$ and $f_n \in L^\infty(G)$, then $\sum_{i=1}^n a_i (x_i f_i - f_i) \in \mathcal{A}_0$ where $a_n$ is a nonzero constant. Indeed, let $\varepsilon > 0$.

Since $x_n f_n - f_n \in \mathcal{A}_0$, there are $\lambda_k > 0$, $y_k \in G$ ($k = 1, 2, \ldots, N$) such that $\sum_{k=1}^N \lambda_k = 1$ and

$$\left\| \sum_{k=1}^N \lambda_k y_k (x_n f_n - f_n) \right\|_\infty < \frac{\varepsilon}{2 |a_n|}.$$ 

Since $\sum_{k=1}^N \lambda_k y_k F \in \mathcal{A}_0$, there are $w_l > 0$ and $z_l \in G$ ($l = 1, 2, \ldots, L$) such that $\sum_{l=1}^L w_l = 1$ and

$$\left\| \sum_{l=1}^L w_l z_l \left( \sum_{k=1}^N \lambda_k y_k F \right) \right\|_\infty < \frac{\varepsilon}{2}.$$ 

Hence

$$\left\| \sum_{l=1}^L w_l z_l \left( \sum_{k=1}^N \lambda_k y_k \left( \sum_{i=1}^n a_i (x_i f_i - f_i) \right) \right) \right\|_\infty$$

$$\leq \left\| \sum_{l=1}^L w_l z_l \left( \sum_{k=1}^N \lambda_k y_k F \right) \right\|_\infty + |a_n| \sum_{l=1}^L w_l \left\| \sum_{k=1}^N \lambda_k y_k (x_n f_n - f_n) \right\|_\infty$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

By induction, $\mathcal{A} \supseteq \text{span}\{x f - f : x \in G, f \in L^\infty(G)\}$. $\square$
**Theorem C.** For a locally compact group $G$, $G$ is amenable as a discrete group if and only if $\sum_{i=1}^{n} \lambda_i x_i f \in A_0$ for any $f \in A_0$, $\lambda_i > 0$, and $x_i \in G$ with $\sum_{i=1}^{n} \lambda_i = 1$.

**Proof.** This is a direct consequence of Theorem 2.3 of [2] and Lemma B. □

The following theorem confirms the conjecture of Rosenblatt and Yang [3]:

**Theorem D.** If $A \subseteq A_R$, then $G$ is amenable as a discrete group.

**Proof.** By Theorem C, it suffices to show that $\sum_{i=1}^{n} \lambda_i x_i f \in A_0$ for any $f \in A_0$, $x_i \in G$, and $\lambda_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \lambda_i = 1$. Since $A_0 \subseteq A \subseteq A_R$, $f \in A_R$ and it right averages to 0 by Corollary 1.4 of [3]. Hence for any $\varepsilon > 0$, there are $\beta_k > 0$, $y_k \in G$ $(k = 1, 2, \ldots, N)$ such that $\sum_{k=1}^{N} \beta_k = 1$ and $\sum_{k=1}^{N} \beta_k y_k \|_{\infty} < \varepsilon$. Not that for each $i = 1, 2, \ldots, n$,

$$\left\| x_i \left( \sum_{k=1}^{N} \beta_k y_k \right) \right\|_{\infty} = \left\| \sum_{k=1}^{N} \beta_k (x_i f) y_k \right\|_{\infty} < \varepsilon.$$ 

Hence

$$\left\| \sum_{k=1}^{N} \beta_k \left( \sum_{i=1}^{n} \lambda_i x_i f \right) y_k \right\|_{\infty} \leq \sum_{i=1}^{n} \lambda_i \left\| \sum_{k=1}^{N} \beta_k (x_i f) y_k \right\|_{\infty} < \varepsilon,$$

that is $\sum_{i=1}^{n} \lambda_i x_i f$ right averages to 0. By Lemma A, $\sum_{i=1}^{n} \lambda_i x_i f \in A$. Hence $\sum_{i=1}^{n} \lambda_i x_i f$ left averages to 0 by Corollary 1.4 of [3] again. Therefore, $\sum_{i=1}^{n} \lambda_i x_i f \in A_0$. □

**References**