

KOSZUL HOMOLOGY OF COHEN-MACAULAY RINGS WITH LINEAR RESOLUTIONS

CARLOS RENTERÍA AND RAFAEL H. VILLARREAL

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ABSTRACT. The first Koszul homology module of a Cohen-Macaulay ideal with a linear resolution is studied and some new examples of rings of minimal multiplicity are presented.

1. INTRODUCTION

Let R be a polynomial ring over a field k , and let I be a graded ideal of R . The algebra $S = R/I$ is *Cohen-Macaulay* (C-M for short) if the projective dimension of S as an R -module is equal to the height of I . The ideal I has a *p-linear resolution* if I is generated by forms of degree p and if all the maps of its graded minimal resolution by free R -modules have linear entries. We say that S has a *p-linear resolution* if I does, and that I is C-M if S is.

Examples of Cohen-Macaulay algebras with linear resolutions include rings of minimal multiplicity [17], the coordinate ring of a variety defined by the submaximal minors of a generic symmetric matrix [13], the coordinate ring of a variety defined by the maximal minors of a generic matrix [3], and some face rings [5].

Cohen-Macaulay rings with linear resolutions have been studied by Sally [17] for the case $p = 2$, and by Schenzel [18] for the general case; more general rings with linear resolutions have been examined in [21, 9, 4].

In this work we use some of the techniques introduced by Kustin, Miller, and Ulrich [14], and by Vasconcelos [22] to study the Koszul homology of Cohen-Macaulay ideals with linear resolution.

We now describe the contents of this paper. In §2 we consider a C-M ideal I of height g with a p -linear resolution. If $g = 2$, Avramov and Herzog [1] have shown that the Koszul homology of I is Cohen-Macaulay. We are able to prove that if I is generically a complete intersection satisfying $g \geq 3$ and $p \geq 2$ then the first Koszul homology module of I is not C-M.

In §3 we present a somewhat different proof of a result due to Cavaliere, Rossi, and Valla [2] which characterizes C-M rings with linear resolution; such

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rings were described in [18] as the class of extremal rings. Finally we show some new examples of face rings of minimal multiplicity.

The basic reference for the definitions and notation on combinatorics and commutative rings will be [19, 15].

2. NUMERICAL STUDY OF KOSZUL HOMOLOGY

Let us recall some facts and fix some notation that will be used throughout this paper. The main reference for Koszul homology is [11].

Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a polynomial ring over a field k , with its usual graduation. Let I be a graded ideal of R . By the resolution of $S = R/I$, we mean the minimal graded resolution of S by free R -modules:

$$(1) \quad 0 \rightarrow \bigoplus_{i=1}^{b_g} R(-d_{gi}) \rightarrow \cdots \rightarrow \bigoplus_{i=1}^{b_1} R(-d_{1i}) \rightarrow R \rightarrow S = R/I \rightarrow 0.$$

The integers b_1, \dots, b_g are the Betti numbers of S . The letters in $()$ are the twists; they indicate a shift in the graduation, e.g., $R(d)_i = R_{d+i}$. The ideal I has a *pure resolution* if there are constants $d_1 < d_2 < \cdots < d_g$ with $d_{1i} = d_1, \dots, d_{gi} = d_g$ for all i . If, in addition, $d_i = d_1 + i - 1$ for $2 \leq i \leq g$, the resolution is said to be *d_1 -linear*.

The ideal I is *generically a complete intersection* if I is unmixed and the localizations of I at its minimal primes are complete intersections. Set d equal to $\dim S$, the Krull dimension of S . A set $\mathbf{z} = \{z_1, \dots, z_d\}$ of homogeneous elements of $S_+ = \bigoplus_{i=1}^{\infty} S_i$ is a *system of parameters* (which we will abbreviate h.s.o.p) for S if $\text{rad}(\mathbf{z}) = S_+$.

Let $M = \bigoplus_{i=0}^{\infty} M_i$ be a finitely generated graded module over R ; its Hilbert function and Hilbert series are defined by $H(M, i) = \dim_k(M_i)$ and $F(M, z) = \sum_{i=0}^{\infty} H(M, i)z^i$, respectively. The length of an Artinian module M will be denoted by $l(M)$.

The graded version of the following criterion plays an important role in the study of the first Koszul homology module of a Cohen-Macaulay ideal.

Proposition 2.1 ([7]). *Let S be a Cohen-Macaulay local ring and let M be a finitely generated S -module with a well-defined and positive rank, and let \mathbf{y} be a system of parameters for S . Then*

$$l(S/(\mathbf{y})) \cdot \text{rank}(M) \leq l(M/(\mathbf{y})M).$$

Furthermore equality holds if and only if M is Cohen-Macaulay.

Our main result is

Theorem 2.1. *Let I be a graded ideal of R . Assume I is generically a complete intersection and that I is a Cohen-Macaulay ideal of height g with a p -linear resolution*

$$0 \rightarrow R^{b_g}(-(p+g-1)) \rightarrow \cdots \rightarrow R^{b_k}(-(p+k-1)) \rightarrow \cdots \rightarrow R^{b_1}(-p) \rightarrow I \rightarrow 0.$$

If $p \geq 2$ and $g \geq 3$, then $H_1(I)$, the first Koszul homology module of I , is not Cohen-Macaulay.

Proof. We set $S = R/I$. The module $H_1 = H_1(I)$ has a well-defined rank equal to $b_1 - g$. Thus by Proposition 2.1, it suffices to prove that for some system of parameters \mathbf{y} , $l(H_1(I)/\mathbf{y}H_1(I)) > \text{rank}(H_1(I)) \cdot l(S/\mathbf{y}S)$.

We start by making a specialization to the case of a polynomial ring of dimension g . Since we may assume that k is infinite, there exists a system of parameters $\mathbf{y} = \{y_1, \dots, y_d\}$ for S with each y_i a form of degree one of R . We make two observations: (i) Because $\text{Tor}_i(S, R/(\mathbf{y})) = 0$ for $i \geq 1$, it is clear that the minimal resolution of $\bar{S} = S/(\mathbf{y}S)$ as an $\bar{R} (= R/(\mathbf{y}))$ -module has the same twists and Betti numbers as the minimal resolution of S over R . (ii) With H_1 Cohen-Macaulay it is easy to see that the first Koszul homology module of $I \otimes \bar{R}$ over \bar{R} is precisely $H_1(I) \otimes \bar{R}$. We may then in the sequel assume that S is zero-dimensional.

We will complete the integer $r = \text{rank}(\text{original } H_1(I)) \cdot l(S)$, with partial Hilbert sums contributing to $l(H_1)$. We first notice that the length of S and its Betti numbers can be calculated using the results of [8, 12]:

$$(2) \quad b_k = \binom{p+k-2}{k-1} \cdot \binom{p+g-1}{g-k}, \quad \text{and } l(S) = \binom{p+g-1}{g}.$$

From the Koszul complex we obtain the exact sequences

$$\begin{aligned} 0 \rightarrow B_1 \rightarrow Z_1 \rightarrow H_1 \rightarrow 0, \\ 0 \rightarrow Z_2 \rightarrow R^{\binom{b_1}{2}}(-2p) \rightarrow B_1 \rightarrow 0, \end{aligned}$$

where Z_i and B_i are the modules of cycles and boundaries defining $H_i(I)$. To simplify notation we set $l(M)_i = H(M, i)$, the dimension of the i th component of the graded module M . We may write

$$l(H_1)_i = l(Z_1)_i - l(B_1)_i = l(Z_1)_i - \binom{b_1}{2} l(R(-2p))_i + l(Z_2)_i,$$

and therefore

$$(3) \quad l(H_1) \geq \sum_{i=0}^{2p} l(H_1)_i \geq \sum_{i=0}^{2p} l(Z_1)_i - \binom{b_1}{2} \sum_{i=0}^{2p} l(R(-2p))_i.$$

From the minimal resolution of S we obtain

$$l(S)_i = l(R)_i - b_1 l(R(-p))_i + l(Z_1)_i \geq 0.$$

Hence $l(Z_1)_i \geq b_1 l(R(-p))_i - l(R)_i$ and $l(Z_1)_i = 0$ for $0 \leq i \leq p$, which leads to

$$(4) \quad \sum_{i=0}^{2p} l(Z_1)_i \geq b_1 \sum_{i=p+1}^{2p} l(R(-p))_i - \sum_{i=p+1}^{2p} l(R)_i.$$

Finally by (3) and (4) we have

$$(5) \quad l(H_1) \geq b_1 \sum_{i=p+1}^{2p} l(R(-p))_i - \sum_{i=p+1}^{2p} l(R)_i - \binom{b_1}{2}.$$

The proof now reduces to showing that the right-hand side of (5) is greater than $\text{rank}(H_1(I)) \cdot l(S) = (b_1 - g) \cdot l(S)$, that is, we must show that the following inequality holds for $p \geq 2$ and $g \geq 3$,

$$f(p, g) = b_1 \sum_{i=p+1}^{2p} l(R(-p))_i - \sum_{i=p+1}^{2p} l(R)_i - \binom{b_1}{2} - (b_1 - g) \cdot l(S) > 0.$$

It is not hard to see that $f(p, g)$ simplifies to

$$f(p, g) = \binom{b_1 + 1}{2} + (1 + g) \binom{p + g - 1}{p - 1} - \binom{2p + g}{g}.$$

Observe that from this equality we obtain the inequality:

$$2f(p, g) > \binom{p + g - 1}{p}^2 - 2 \binom{2p + g}{g}.$$

It is easy to check that $f(p, g) > 0$ for $g \in \{3, 4, 5\}$ and $p \geq 2$. The required inequality is now a direct consequence of Lemma 2.1.

Lemma 2.1. *Let p and g be positive integers and let*

$$\psi(p) = \binom{p + g - 1}{g - 1}^2 - 2 \binom{2p + g}{g}.$$

If $p \geq 2$ and $g \geq 6$ then $\psi(p) > 0$.

Proof. Notice the equality $6\psi(2) = (g + 1)(g^3 - 3g^2 - 13g - 12)$, which is certainly positive for $g \geq 6$. We proceed by induction on p . Assume $\psi(p) > 0$. It is easy to check that $\psi(p + 1)$ is greater than

$$2 \binom{2p + g}{g} \left[\frac{(p + 1)(4(g - 2)p^2 + (3g^2 - 3g - 8)p + g^2 - 3g - 2)}{(p + 1)^2(2p + 1)(2p + 2)} \right].$$

Since the right-hand side of this inequality is positive for $p \geq 2$ and $g \geq 3$, the induction step is complete. \square

3. COHEN-MACAULAY RINGS WITH LINEAR RESOLUTIONS

The next result, when applied to a Cohen-Macaulay ideal I generated by forms of the same degree, provides a bound for the minimum number of generators of I .

Proposition 3.1. *Let R be a polynomial ring over a field k . Let $I = \bigoplus_{i=p}^{\infty} I_i$ be a graded ideal in R of height g with $I_p \neq (0)$. If I is Cohen-Macaulay then*

$$H(I, p) \leq \binom{p + g - 1}{p}.$$

Proof. We may assume that k is infinite, since a change of the coefficient field can be easily carried out using the functor $(\cdot) \otimes_k K$, where K is an infinite field extension of k . This change of the coefficient field preserves the hypothesis on I and leaves both the height of I and the dimensions of the vector spaces of forms of a given degree in I unchanged. We set $S = R/I$. There is a h.s.o.p $\mathbf{y} = \{y_1, \dots, y_d\}$ for S , where each y_i is a linear form in R . If we tensor the minimal resolution of S with $\bar{R} = R/(\mathbf{y})$ it follows that $0 \leq H(\bar{S}, p) = H(\bar{R}, p) - H(I, p)$, where $\bar{S} = S/(\mathbf{y})S$. To get the desired inequality notice that \bar{R} is a polynomial ring in g variables over the field k . \square

Let I be a graded ideal of R and let $S = R/I$. The Hilbert-Serre theorem asserts that there is a (unique) polynomial $h(z) = h_0 + h_1 z + \dots + h_r z^r$ with integral coefficients so that $h(1) \neq 0$ and satisfying

$$F(S, z) = \frac{h(z)}{(1 - z)^d},$$

where $d = \dim S$. The h -vector of S is $h(S) = (h_0, \dots, h_r)$. If S is an Artinian algebra the socle of S is given by $\text{Soc}(S) = (0 :_S S_+)$.

Theorem 3.1 ([20]). *Let S be a finitely generated graded algebra over a field k . Let $\mathbf{y} = \{y_1, \dots, y_d\}$ be a homogeneous system of parameters for S with $a_i = \deg(y_i)$. Then S is Cohen-Macaulay if and only if*

$$F(S, z) = F(\bar{S}, z) \left/ \prod_{i=1}^d (1 - z^{a_i}), \right.$$

where \bar{S} is the quotient ring $S/(\mathbf{y})$ with its natural grading.

Our proof of the next result uses essentially the theorem above and a well-known description of the socle of an Artinian algebra.

Theorem 3.2 ([2]). *Let R be a polynomial ring over a field k . Let $I = \bigoplus_{i=p}^{\infty} I_i$ be a graded ideal in R of height g with $I_p \neq (0)$. If I is a Cohen-Macaulay ideal, then $S = R/I$ has a p -linear resolution if and only if the following equality holds*

$$H(I, p) = \binom{p + g - 1}{p}.$$

Proof. Let (1) be the minimal resolution of $S = R/I$. We order the shifts so that $d_{k1} \leq \dots \leq d_{kb_k}$ for $k = 1, \dots, g$; by the minimality of the resolution we have $d_{11} < d_{21} < \dots < d_{g1}$. Observe that if the resolution of S is linear then (2) gives the required equality.

Conversely assume the equality above. Since we may assume that k is infinite, there is a h.s.o.p $\mathbf{y} = \{y_1, \dots, y_d\}$ for S , where each y_i is a form of degree 1 in R . Set $\bar{S} = S/(\mathbf{y})S$ and $\bar{R} = R/(\mathbf{y})$. From Theorem 3.1 we derive that the h -vector of S satisfies $h_p = 0$, and also get $h_i = H(\bar{S}, i)$. Therefore $h_i = 0$ for all $i \geq p$. If we use [6, p. 131], we obtain a degree 0 isomorphism

$$\text{Soc}(\bar{S}) \cong \bigoplus_{i=1}^{b_g} k[g - d_{gi}]$$

of graded k -vector spaces, in particular the socle of \bar{S} can only live in degrees $d_{gi} - g$, hence we conclude the inequality $d_{gi} - g \leq p - 1$. On the other hand, the minimality of the resolution of \bar{S} gives $p + (g - 1) \leq d_{gi}$ for all i . Altogether we have $d_{gi} = p + g - 1$. Because I is C-M, then by [6, §4] we must have $p \leq d_{1b_1} < d_{2b_2} < \dots < d_{gb_g} = p + g - 1$, which implies that the resolution is p -linear, as required. \square

Remark. In the proof above the condition “ $h_i = 0$ for all $i \geq p$ ” readily implies that S is an extremal ring, and therefore by [18, Theorem A] S has a linear resolution.

As an application we will present in Example 3.1 a family of C-M face rings with 2-linear resolution, but first we need to introduce some notation. Let \mathcal{G} be a graph on the vertex set $\mathcal{V} = \{x_1, \dots, x_n\}$. Given a field k the ideal $I(\mathcal{G})$ associated to \mathcal{G} [23] is defined as the ideal of the polynomial ring $R = k[x_1, \dots, x_n]$ generated by the set of monomials $x_i x_j$ such that x_i is adjacent to x_j . If all the vertices of \mathcal{G} are isolated we set $I(\mathcal{G}) = (0)$. A graph \mathcal{G} is called Cohen-Macaulay if $R/I(\mathcal{G})$ is a C-M ring. Notice that if \mathcal{G}

is a C-M graph with q edges then by Proposition 3.1 we have $q \leq g(g+1)/2$, where g is the height of $I(\mathcal{G})$.

Definition 3.1. Let \mathcal{G} be a graph with q edges. \mathcal{G} is called a saturated graph if \mathcal{G} is Cohen-Macaulay and $q = g(g+1)/2$, where g is the height of $I(\mathcal{G})$.

Remark. Let \mathcal{G} be a graph with connected components $\mathcal{G}_1, \dots, \mathcal{G}_m$. Then using [23, Lemma 4.1] it is readily seen that \mathcal{G} is a saturated graph if and only if \mathcal{G}_i is saturated for all i . Moreover if \mathcal{G} is a saturated graph then at most one of its connected components has more than one vertex.

Corollary 3.1. Let \mathcal{G} be a Cohen-Macaulay graph over a field k . Then \mathcal{G} is a saturated graph if and only if $R/I(\mathcal{G})$ has a 2-linear resolution.

The following example was studied in [23]. It is a particular case of a family of C-M graphs containing all C-M trees.

Example 3.1. Let \mathcal{G} be a graph with vertex set $\mathcal{V} = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ and edge set $\mathcal{E} = \{\{x_k, y_k\}, \{y_i, y_j\} | k = 1, \dots, n \text{ and } 1 \leq i < j \leq n\}$. Then \mathcal{G} is a saturated graph over any field k .

Definition 3.2. Let I be an ideal of $R = k[x_1, \dots, x_n]$ generated by square free monomials. The ring $S = R/I$ is called a face ring. The Reisner-Stanley simplicial complex Δ associated to I has vertex set $\mathcal{V} = \{x_1, \dots, x_n\}$ and its faces are defined by

$$\Delta = \{\{x_{i_1}, \dots, x_{i_k}\} | i_1 < \dots < i_k, x_{i_1} \cdots x_{i_k} \notin I\}.$$

S is also called the Reisner-Stanley ring of Δ , and is denoted by $k[\Delta]$.

In [5] Fröberg characterized monomial ideals with p -linear resolutions via simplicial homology. Herzog and Kühn have also studied such ideals, and in [8] they have established a simple criterion to determine when ideals generated by monomials have a linear resolution. As a consequence of the main result of [5] or by [8, Proposition 4] the following family of ideals with linear resolutions is obtained.

Example 3.2 ([5]). Let Δ be the skeleton of dimension $p-2$ of a simplex of dimension $n-1$, $p \geq 2$. Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k . Then the corresponding face ring

$$k[\Delta] = R/(\{x_{i_1} \cdots x_{i_p} | 1 \leq i_1 < \dots < i_p \leq n\})$$

has a p -linear resolution.

Remark. In the example above the height of I_Δ is $n-p+1$. Since I_Δ is a C-M ideal by [10, Example A], Theorem 3.2 applies.

Examples. (a) Let Δ be a simplicial complex of dimension d on the vertex set $\mathcal{V} = \{x_1, \dots, x_n\}$. Notice that if $d = 0$ then $I_\Delta = (x_i x_j; 1 \leq i < j \leq n)$ is a C-M ideal and therefore by Theorem 3.2, $k[\Delta]$ has a 2-linear resolution. Assume Δ is connected and $\dim \Delta = 1$. If Δ is the complete graph on its vertices then $I_\Delta = (x_i x_j x_k; 1 \leq i < j < k \leq n)$ is the 1-skeleton of a simplex and $k[\Delta]$ has a 3-linear resolution. If Δ is not the complete graph Theorem 3.2 shows that $k[\Delta]$ has a 2-linear resolution if and only if Δ is a tree. For a complete description of face rings of Krull dimension 2 with linear resolutions see [5, Theorem 10].

(b) Let $R = k[a, \dots, f]$ be a polynomial ring over a field k . The face ring $S = R/I$, where $I = (abc, abd, ace, adf, aef, bcf, bde, bef, cde, cdf)$ was studied by Reisner. In [16] S is shown to be C-M if k has characteristic other than 2 and non-C-M otherwise. Thus in the first case by Theorem 3.2, S has a 3-linear resolution (cf. [5, Example 3]).

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INSTITUTO POLITÉCNICO NACIONAL, ESCUELA SUPERIOR DE FÍSICA Y MATEMÁTICAS, UNIDAD ADOLFO LÓPEZ MATEOS, 07300 MÉXICO, D.F.