THE FLAT STRIP THEOREM FAILS
FOR SURFACES WITH NO CONJUGATE POINTS

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(Communicated by Jonathan M. Rosenberg)

Abstract. A compact $C^\infty$ surface with no conjugate points is constructed so that there are two homotopic closed geodesics that do not bound a flat annulus.

One of the basic rigidity properties of manifolds with nonpositive curvature is the

Flat Strip Theorem [EON]. Let $\beta$ and $\gamma$ be geodesics in a simply connected manifold with nonpositive curvature. Suppose that $\beta$ and $\gamma$ have finite Hausdorff distance. Then $\beta$ and $\gamma$ are the edges of a flat strip, i.e., an isometrically and totally geodesically embedded copy of $I \times \mathbb{R}$.

Two natural generalizations of nonpositive curvature are the no focal point and no conjugate point properties. These can be described in terms of Jacobi fields [OS1, Proposition 4]. A complete Riemannian manifold has

(i) nonpositive curvature if and only if $(Y, Y)'' \geq 0$ for every Jacobi field $Y$;
(ii) no focal points if and only if $(Y, Y)'(t) > 0$ whenever $t > 0$ and $Y$ is a nontrivial Jacobi field with $Y(0) = 0$;
(iii) no conjugate points if and only if $(Y, Y)(t) > 0$ whenever $t > 0$ and $Y$ is a nontrivial Jacobi field with $Y(0) = 0$.

Gulliver [G] has given examples that show (iii) $\nRightarrow$ (ii) $\nRightarrow$ (i).

The flat strip theorem holds for manifolds with no focal points [OS2, E]. The present paper shows that it fails for surfaces with no conjugate points. We construct a compact $C^\infty$ surface $S$ with no conjugate points that contains a closed annulus that is foliated by homotopic closed geodesics, but is not flat. A lift of this annulus to the universal cover is a nonflat strip bounded by two geodesics with finite Hausdorff distance.

Here is a brief description of $S$. Consider a torus of revolution $T$ that is flat except for one small bulge. The torus $T$ is foliated by closed meridian...
geodesics. Let \( S_1 \) be a thin annulus that is a union of meridians. A \( C^1 \) approximation to \( S \) can be constructed by taking a surface with constant curvature \(-1\) that contains a simple closed geodesic with the same length as the meridians of \( T \), cutting the surface along this geodesic and then gluing in \( S_1 \). One could think of \( S \) as being obtained from this surface by carefully smoothing the metric near \( \partial S_1 \). Instead we construct a \( C^\infty \) Riemannian metric \( g \) on \( S^1 \times \mathbb{R} \) so that

(i) the curve \( \sigma = S^1 \times \{0\} \) is a geodesic;
(ii) the natural coordinates \( s \in S^1 \) and \( y \in \mathbb{R} \) are Fermi coordinates for \( \sigma \);
(iii) \( \sigma \) has a neighbourhood \( \Sigma \) isometric to \( S_1 \);
(iv) the curvature is \(-1\) outside a neighbourhood of \( \Sigma \).

The two ends of \((S^1 \times \mathbb{R}, g)\) are then replaced by compact ends with curvature \(-1\), using a standard construction. Property (ii) of \( g \) makes it easy to prescribe the curvature near \( \partial \Sigma \). Note finally that \( S \) has focal points because there are periodic Jacobi fields along \( \sigma \).

1. Preliminaries

Suppose we define a metric on a surface with respect to the coordinates \( s \) and \( y \) by

\[
\begin{pmatrix}
g_{ss} & g_{sy} \\
g_{ys} & g_{yy}
\end{pmatrix}
= 
\begin{pmatrix}
\psi^2(s, y) & 0 \\
0 & 1
\end{pmatrix},
\]

where \( \psi > 0 \) and \( \psi \) is the solution of the initial value problem

\[
(1.2) \quad \psi_{yy}(s, y) + K(s, y)\psi(s, y) = 0, \quad \psi(s, 0) = 1, \quad \psi_y(s, 0) = 0.
\]

Then the curve \( \sigma \) defined by \( y = 0 \) is a geodesic for which \( s \) and \( y \) are Fermi coordinates; \( s \) measures arclength along \( \sigma \) and \( |y| \) is distance from \( \sigma \). The curvature at the point with coordinates \( (s, y) \) is \( K(s, y) \). In our case, \( \sigma \) will be a closed geodesic of length \( 2\pi \), so we take \( s \in S^1 \). If \( s_1, s_2 \in S^1 \), \((s_1, s_2)\) and \([s_1, s_2]\) will denote the open and closed arcs respectively from \( s_1 \) to \( s_2 \) in the anticlockwise direction.

2. Construction of \( T \) and \( S \)

Let \( \alpha = 1/110 \). The torus \( T \) will be a warped product \( S^1 \times_f S^1 \) (warped products are described in \[BON\]), where \( f \) is a \( C^\infty \) function on \( S^1 \) such that \( f(s) = 1 \) if \( s \in [e^{i\alpha}, e^{-i\alpha}] \) and \( 1 < f(s) < 9 \) if \( s \in (e^{-i\alpha}, e^{i\alpha}) \). Thus \( T \) is a torus of revolution with the curves \( S^1 \times_f \{s\}, s \in S^1 \), as meridian geodesics. Let \( \sigma_1 \) be the geodesic \( S^1 \times_f \{1\} \). We choose \( f \) so that the curvature \( K_T \) of \( T \) satisfies \( |K_T| \leq 10^{-6} \) throughout \( T \). It is clear from this that there is a diffeomorphism \( \pi : S^1 \times [-1/10, 1/10] \to \{p \in T : \text{dist}(p, \sigma_1) \leq 1/10\} \) such that \((s, y)\) are the Fermi coordinates with respect to \( \sigma_1 \) for the point \( \pi(s, y) \).

We now choose the curvature \( K(s, y) \) that we want for the metric \( g \) on \( S^1 \times \mathbb{R} \). Choose a \( C^\infty \) function \( h : S^1 \to [\alpha, 9\alpha] \) such that

\[
h(s) = \begin{cases} 
9\alpha & \text{if } s \in [e^{-i\alpha}, e^{i\alpha}], \\
\alpha & \text{if } s \in [e^{2i\alpha}, e^{-2i\alpha}].
\end{cases}
\]
Let $H = \{(s, y) \in S^1 \times \mathbb{R} : |y| \leq h(s)\}$ and $H_\alpha = \{(s, y) \in S^1 \times \mathbb{R} : |y| \leq h(s) + \alpha\}$. Choose a $C^\infty$ function $\theta : S^1 \times \mathbb{R} \to [0, 1]$ with $\theta = 1$ on $H$ and $\theta = 0$ outside $H_\alpha$. Define $K(s, y)$ by

$$K(s, y) = \begin{cases} K_T(\pi(s, y)) & \text{if } (s, y) \in H, \\ \theta(s, y)K_T(\pi(s, y)) - \{1 - \theta(s, y)\} & \text{if } (s, y) \in H_\alpha \setminus H, \\ -1 & \text{if } (s, y) \notin H_\alpha. \end{cases}$$

See Figure 1. Observe that $K_T$ vanishes in the region $[e^{i\alpha}, e^{-i\alpha}] \times f S^1$ and

$$(2.1) \quad -1 \leq K \leq 10^{-6} \quad \text{everywhere;}$$
$$(2.2) \quad K \leq 0 \quad \text{outside } (e^{-i\alpha}, e^{i\alpha}) \times (-10\alpha, 10\alpha);$$
$$(2.3) \quad K = -1 \quad \text{outside } ((e^{-2i\alpha}, e^{2i\alpha}) \times (-10\alpha, 10\alpha)) \\
\quad \cup (S^1 \times (-2\alpha, 2\alpha)).$$

Finally $g$ is the metric represented with respect to the $s$-$y$ coordinates by the matrix (1.1) in which $\psi$ satisfies (1.2). It is easily shown using (2.1) and
that \( \psi(s, y) > 0 \) for all \((s, y)\). One can even see that \( \psi(s, y) \to \infty \)
uniformly as \(|y| \to \infty\). The construction ensures that the curve \( \sigma(t) = (e^{it}, 0) \)
is a unit speed geodesic for the metric \( g \) and \((s, y)\) are Fermi coordinates for \( \sigma \)
throughout \( \mathbb{S}^1 \times \mathbb{R} \).

Let \( \Sigma_1 = \mathbb{S}^1 \times_f [e^{-i\alpha}, e^{i\alpha}] \), so \( \Sigma_1 \) is a neighbourhood of \( \sigma_1 \) foliated by meridian geodesics. It is clear from the choice of \( f \) and \( g \) that \( \Sigma_1 \subseteq \pi(H) \).

Since \( K = K_T \circ \pi \) on \( H \), we see that \( \Sigma \overset{\text{def}}{=} \pi^{-1}(\Sigma_1) \) is isometric to \( \Sigma_1 \).

We now outline the procedure for replacing the ends of \((S^1 \times \mathbb{R}, g)\) with compact ends. It follows a similar construction in [BBB]. If we cut a component of \((S^1 \times \mathbb{R}) \setminus N\) along the geodesic \( \{1\} \times \mathbb{R} \) we obtain a fan-shaped subset of the Poincaré disc shown in Figure 2.

This subset is bounded by two geodesic rays \( \eta \) and \( \eta' \) and a curve corresponding to a component of \( \partial N \). For any large enough \( n \), it is possible to draw a sequence of \( 4n + 1 \) hyperbolic geodesic segments \( d_n', c_n', d_{n-1}' \), \ldots, \( d_1', c_1', d_0', c_0', d_1'' \), \ldots, \( c_n'' \) such that

- \( d_n' \) and \( d_n'' \) are orthogonal to \( \eta' \) and \( \eta'' \) respectively;
- adjacent geodesics in the sequence are orthogonal;
- \( c_i' \) and \( c_i'' \) have the same length for \( 1 \leq i \leq n \).

By identifying \( \eta' \) with \( \eta'' \) and \( c_i' \) with \( c_i'' \), \( 1 \leq i \leq n \), we obtain a hyperbolic metric on the sphere with \( n + 2 \) punctures. One of the holes is bounded by a copy of a component of \( \partial N \) and the others by closed geodesics \( d_0, \ldots, d_n \).

Now we can adjoin handles with curvature \(-1\) along \( d_0, \ldots, d_n \).

**Figure 2**
3. Geodesics in $N$

Let $N = S^1 \times [-1/10, 1/10]$. Note that $H_\alpha \subseteq N$, since $10\alpha < 1/10$.

3.1. Lemma. For any $(s, y) \in N$ and any unit vector $u \in T_{(s, y)}N$,

(i) $0.99 \leq \|\frac{\partial}{\partial s}(s, y)\| \leq 1.01$;

(ii) $\|\nabla \frac{\partial}{\partial y}\| \leq 0.11$.

Proof. Note that $|K(s, y)| \leq 1$, by (2.1). Since $|y| \leq 1/10$, it follows from (1.2) and the Sturm Comparison Theorem that

$0.99 \leq \cos(1/10) \leq \psi(s, y) \leq \cosh(1/10) \leq 1.01$.

This proves (i), since $\psi = \|\frac{\partial}{\partial s}\|$.

Since the level curves for $s$ are the geodesics orthogonal to $\sigma$ and $\frac{\partial}{\partial y}(s, 0)$ is a smooth field of unit vectors normal to $\sigma$,

$\nabla \frac{\partial}{\partial y}(s, y) = 0$ and $\nabla \frac{\partial}{\partial y}(s, 0) = \nabla \frac{\partial}{\partial y}(s, 0) = 0$.

Since $\frac{\partial}{\partial s}$ is a Jacobi field along each geodesic $s = \text{const}$,

$\frac{\partial}{\partial s} \frac{\partial}{\partial y} = \frac{\partial}{\partial y} \frac{\partial}{\partial s} = -R(\frac{\partial}{\partial s}, \frac{\partial}{\partial y}) \frac{\partial}{\partial y}$,

where $R$ is the curvature tensor for $g$. Thus if $u = a\frac{\partial}{\partial s}(s, y) + b\frac{\partial}{\partial y}(s, y)$, we have

$\|\nabla u\| = \|a\nabla \frac{\partial}{\partial s}\| \leq |a| \int_{0}^{y} \|R(\frac{\partial}{\partial s}, \frac{\partial}{\partial y})\frac{\partial}{\partial y}(s, z)\| dz$.

Since $\langle R(\frac{\partial}{\partial s}, \frac{\partial}{\partial y}) \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle = 0$ and $\|\frac{\partial}{\partial y}\| = 1$,

$\|R(\frac{\partial}{\partial s}, \frac{\partial}{\partial y}) \frac{\partial}{\partial y}\| = \|\frac{\partial}{\partial s}(s, y)\| \cdot |K(s, y)| \leq \|\frac{\partial}{\partial s}(s, y)\|$.

Thus $\|\nabla u\| \leq |a| \cdot |y| \cdot \sup_{N} \|\frac{\partial}{\partial s}(s, y)\| \leq (0.99)^{-1} \cdot (1/10) \cdot 0.11 < 0.11$, by (i). □

3.2. Corollary. Let $\gamma$ be a maximal unit speed geodesic in $N$ that makes angle $\pi/4$ with the $s$ and $y$ directions at $\gamma(0)$. Then the angle between $\gamma$ and the $s$ direction is always between $\pi/6$ and $\pi/3$.

Proof. Since $\frac{\partial}{\partial y}$ is the unit vector field in the $s$-direction, the cosine of this angle is $c(t) = \langle \dot{\gamma}(t), \frac{\partial}{\partial y}(\gamma(t)) \rangle$. Because $N$ is symmetric about $\sigma$, we can assume that $c(0) = 1/\sqrt{2}$, not $-1/\sqrt{2}$. Since $\gamma$ is a geodesic and $\gamma(t) \in N$,

$|c'(t)| = |\langle \dot{\gamma}(t), \nabla_{\gamma(t)} \frac{\partial}{\partial y}(\gamma(t)) \rangle| \leq 0.11$.

Suppose that $-1/2 \leq t_0 \leq t_1 \leq 1/2$ and $\gamma(t)$ is defined for $t_0 \leq t \leq t_1$. If $t \in [t_0, t_1]$,

$\cos(\pi/3) < \cos(\pi/4) - \frac{1}{2} \cdot 0.11 \leq c(t) \leq \cos(\pi/4) + \frac{1}{2} \cdot 0.11 < \cos(\pi/6)$.

We now show that $\gamma$ hits $\partial N$ before $t = 1/2$, in other words that $t_1 < 1/2$. For $0 \leq t \leq t_1$, we have $(y \circ \gamma)'(t) = c(t) > \cos(\pi/3)$. Thus if $t_1 = 1/2$,

$y(\gamma(t_1)) > y(\gamma(0)) + \frac{1}{2} \cos(\pi/3) \geq -1/10 + 1/4 > 1/10$.
which contradicts $\gamma(t_1) \in N$. Similarly $t_0 > -1/2$. □

3.3. **Definition.** A geodesic in $N$ is horizontal (resp. vertical) if the maximal angle that it makes with the $s$-direction (resp. $y$-direction) is at most $\pi/3$.

It follows from Corollary 3.2 that every geodesic in $N$ is horizontal or type $y$ (or both). From Lemma 3.1, we easily obtain

3.4. **Corollary.** Let $\gamma(t)$ be a unit speed geodesic in $N$.

(i) If $\gamma$ is vertical, then $|(y \circ \gamma)'(t)| \geq 1/2$.

(ii) If $\gamma$ is horizontal, then $|(s \circ \gamma)'(t)| > 1/3$.

(iii) If $\gamma$ is horizontal, then $|(y \circ \gamma)'(t)| < 2|(s \circ \gamma)'(t)|$.

4. **Proof that $S$ has no conjugate points**

Proposition 4.4 below will show that along every geodesic $\gamma$ in $S$ the Riccati equation

\[ u'(t) + u^2(t) + K(t) = 0, \]

where $K(t)$ is the curvature of $S$ at $\gamma(t)$, has a solution $u_\gamma(t)$ that is defined for all $t \geq 0$. It then follows that $S$ has no conjugate points. To see this, suppose that $J(t)$ is a Jacobi field along $\gamma$ with $J(a) = 0 = J(b)$ and $J(t) \neq 0$ for $a < t < b$. Then $J(t) = j(t)N(t)$, where $N(t)$ is a continuous field of unit normals along $\gamma$ and $j(t)$ is a solution of the scalar Jacobi equation $j''(t) + K(t)j(t) = 0$. It follows easily that $u = j'/j$ is a continuous solution of (4.1) on the interval $(a, b)$ with $\lim_{t \searrow a} u(t) = \infty$ and $\lim_{t \nearrow b} u(t) = -\infty$, which is impossible since the graph of $u$ would have to cross the graph of $u_\gamma$.

For a fuller discussion, see [BBB, §1].

4.1. **Lemma.** Let $u_i(t)$, $i = 0, 1$, be the solutions of the initial value problems

\[ u_i' + u_i^2 + K_i(t) = 0, \quad u_i(0) = w_i, \quad i = 0, 1. \]

Suppose $w_1 \geq w_0$, $K_1(t) \leq K_0(t)$ for $t \in [0, t_0]$, and $u_0(t_0)$ is defined. Then $u_1(t) \geq u_0(t)$ for $t \in [0, t_0]$.

**Proof.** The difference $\Delta u(t) = u_1(t) - u_0(t)$ satisfies the linear equation

\[ \Delta u' = -(u_0 + u_1)\Delta u + K_0(t) - K_1(t). \] □

4.2. **Lemma.** (i) If $u' + u^2 + 10^{-6} = 0$ and $u(t_0) = 0$, then

\[ u(t) = -10^{-3} \tan 10^{-3}(t - t_0). \]

(ii) If $u' + u^2 - 1 = 0$ and $u(t_0) = -10^{-3}$, then

\[ u(t) = \tanh(t - t_0 - \text{arctanh}(10^{-3})). \]

4.3. **Lemma.** Let $\gamma(t)$ be a geodesic of the torus $T$ defined in §2. Suppose that $\gamma$ crosses one of the parallels of latitude $\{e^{i\alpha}\} \times_f S^1$ or $\{e^{-i\alpha}\} \times_f S^1$ transversally. Then there is a solution of the Riccati equation (4.1) along $\gamma$ that is defined for all $t$ and vanishes whenever $\gamma(t) \in [e^{i\alpha}, e^{-i\alpha}] \times_f S^1$.

Innami [I, Lemma 1.1] has proved essentially the same lemma.

**Proof.** Let $V$ be the Killing field defined by the rotational symmetry. Then $V$ is tangent to the parallels of latitude and has length $f(s)$ on the parallel
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\{s\} \times_f S^1. The Clairaut integral \( \langle \dot{y}(t), V(y(t)) \rangle \) is constant. Since \( ||V|| \) attains its minimum on \( \{e^{ia}\} \times_f S^1 \) and \( \{e^{-ia}\} \times_f S^1 \) and \( y \) is transverse to one of these curves, we have \( \langle \dot{y}(t), V(y(t)) \rangle < ||V(y(t))|| \) for all \( t \). Hence \( y \) is never tangent to \( V \). The restriction of \( V \) to \( y \) is a Jacobi field. Let \( j(t) \) be the component of \( V(y(t)) \) orthogonal to \( y \). Then \( u(t) = \dot{j}(t)/j(t) \) is a solution of the Riccati equation (4.1) along \( y \) which is defined for all \( t \). Since \( f \) is constant on \( [e^{ia}, e^{-ia}] \), it follows that \( j(t) \) is constant and \( u(t) = 0 \) when \( y(t) \in [e^{ia}, e^{-ia}] \times_f S^1 \).

Now consider our surface \( S \). Let \( R = R_- \cup R_0 \cup R_+ \subseteq N \), where

\[
R_- = \{e^{-ia}, e^{ia}\} \times [-10\alpha, -9\alpha], \\
R_0 = \{e^{-ia}, e^{ia}\} \times [-9\alpha, 9\alpha], \\
R_+ = \{e^{-ia}, e^{ia}\} \times [9\alpha, 10\alpha].
\]

See Figure 1. The curvature of \( S \) is nonpositive outside \( R \), by (2.2). For a geodesic \( y \) of \( S \) define inductively a (possibly finite) sequence of times \( t_0, t_1, \ldots \) as follows. Choose \( t_0 \) so that \( t_0 \leq 0 \) and \( y(t_0) \) is not in \( R \) (it is impossible to have \( y(t) \in R \) for all \( t \leq 0 \), cf. Corollary 3.4). For \( i \geq 1 \), let \( t_i \) be the first time after \( t_{i-1} \) when \( y(t) \) enters \( R \); if \( y(t) \) does not enter \( R \) for \( t > t_i \), we set \( t_i = \infty \) and the sequence terminates.

4.4. Proposition. Let \( u_y \) be the solution of (4.1) with \( u_y(t_0) = 0 \). Suppose \( t_i \) is finite, \( u_y(t) \) is defined for \( t_0 \leq t \leq t_i \) and \( u_y(t_i) \geq 0 \). Then \( u_y(t) \) is defined for \( t_i \leq t < t_{i+1} \). If \( t_{i+1} \) is finite, then \( u_y(t_{i+1}) \) is defined and \( u_y(t_{i+1}) \geq 0 \).

Proof. Since \( u'_y(t) < 0 \) if \( |u_y| \) is large, \( u_y(t) \) can fail to be defined for all \( t \geq t_0 \) only if \( u_y(t) \to -\infty \) in finite time. Thus in order to show that \( u_y(t) \) is defined for \( t_i \leq t < t_{i+1} \), it suffices to bound \( u_y(t) \) from below.

Case 0. \( i = 0 \). For \( t_0 \leq t < t_1 \), we have \( K(t) \leq 0 \), since \( y(t) \) is not in \( R \). Using Lemma 4.1 to compare \( u_y(t) \) with the solution of the initial value problem, \( u' + u^2 = 0 \) and \( u(t_0) = 0 \), shows that \( u_y(t) \geq 0 \) for \( t_0 \leq t < t_1 \). Moreover \( u_y(t_1) \geq 0 \) if \( t_1 \) is finite.

When \( i \geq 1 \), let \( T_i \) be the time in \((t_i, t_{i+1})\) when \( y \) leaves \( R \). Since \( y \) does not enter \( R \) for \( T_i < t < t_{i+1} \), the argument of Case 0 shows that it is enough to find a time \( c_i \in (T_i, t_{i+1}) \) such that \( u_y(t) \) is defined for \( t_i \leq t < c_i \) and \( u_y(c_i) \geq 0 \). We study three cases, depending on how \( y \) crosses \( R \) while \( t_i \leq t < t_{i+1} \). See Figure 1. Set \( y(t) = y(y(t)) \) and \( s(t) = s(y(t)) \).

Case 1. \( y \) is vertical. Clearly \( y \) leaves \( N = S^1 \times [-1/10, 1/10] \) while \( t \) is between \( t_i \) and \( t_{i+1} \). Let \( b_i \) be the smallest time such that \( b_i > t_i \) and \( |y(b_i)| = 10\alpha \). By Corollary 3.4, \( b_i - t_i \leq 2|y(b_i) - y(t_i)| \leq 40\alpha \). Recall from (2.1) that \( K(t) \leq 10^{-6} \). Since \( u_y(t_i) \geq 0 \), we see from Lemmas 4.1 and 4.2(i) that, for \( t_i \leq t \leq b_i \),

\[
u_y(t) \geq -10^{-3} \tan^{-1} 10^{-3}(t - t_i) \geq -10^{-3} \tan^{-1} (10^{-3} \cdot 40\alpha) \geq -10^{-3}.
\]

Let \( c_i \) be the first time after \( b_i \) such that \( y(c_i) \in \partial N \). Then \( c_i - b_i \geq 1/10 - 10\alpha = \alpha \), since \( \alpha = 1/110 \). By (2.3), \( K(t) = -1 \) for \( b_i \leq t \leq c_i \). Since \( u_y(b_i) \geq -10^{-3} \), Lemmas 4.1 and 4.2(ii) show that

\[u_y(t) \geq \tanh(t - b_i - \arctanh(10^{-3})) \quad \text{for} \quad b_i \leq t \leq c_i.
\]

Thus \( u_y(t) \) is defined for \( t_i \leq t \leq c_i \) and \( u_y(c_i) \geq \tanh(\alpha - \arctanh(10^{-3})) \geq 0 \).
Case 2. \( \gamma \) is horizontal and does not meet \( R_- \cup R_+ \). Let \( c_i \) be the time in \((t_i, t_{i+1})\) when \( \gamma \) leaves \( R \). We can consider \( \gamma[[t_i, c_i]] \) as a geodesic segment in the torus \( T \) that stretches between the parallels of latitude \( \{e^{-ia}\} \times_f S^1 \) and \( \{e^{ia}\} \times_f S^1 \). Lemma 4.3 shows that there is a solution \( U(t) \) of the Riccati equation along this segment that vanishes at the endpoints. Clearly \( u_y(t) \geq U(t) \) for \( t_i \leq t \leq c_i \) and \( u_y(c_i) \geq 0 \).

Case 3. \( \gamma \) is horizontal and enters \( R_- \cup R_+ \). Since \( N \) is symmetric about the curves \( y = 0 \) and \( s = 1 \), we may assume that \( \gamma \) enters \( R_+ \) and \( s(t) \) moves anticlockwise on \( S^1 \) as \( t \) increases. Let \( b_i \) be the first time after \( t_i \) such that \( s(t) = e^{2ia} \). It is clear from Corollary 3.4(ii) that \( b_i - t_i \leq 3 \text{dist}_{S^1}(s(b_i), s(t_i)) \leq 9\alpha \). As in Case 1, Lemmas 4.1 and 4.2(i) show that, for \( t_i < t < b_i \),

\[
u_y(t) \geq -10^{-3} \tan 10^{-3}(t - t_i) \geq -10^{-3} \tan(10^{-3} \cdot 9\alpha) \geq -10^{-3}.
\]

Choose \( a_i \in [t_i, b_i] \) so that \( \gamma(a_i) \in R_+ \). Then \( s(a_i) \in [e^{-ia}, e^{ia}] \) and, by Corollary 3.4(iii), \( |y(b_i) - y(a_i)| \leq 2 \text{dist}_{S^1}(s(b_i), s(a_i)) \leq 6\alpha \). Since \( y(a_i) \geq 9\alpha \), it follows that \( y(b_i) \geq 3\alpha \).

Let \( c_i = b_i + \alpha \). Then \( y(t) \geq 2\alpha \) for \( b_i \leq t \leq c_i \), when we also have \( s(t) \in [e^{2ia}, e^{3ia}] \). It follows from (2.3) that \( K(t) = -1 \) for \( b_i \leq t \leq c_i \). Since \( u_y(b_i) \geq -10^{-3} \), Lemmas 4.1 and 4.2(ii) show that

\[
u_y(t) \geq \tanh(t - b_i - \arctanh(10^{-3})) \quad \text{for } b_i \leq t \leq c_i.
\]

Thus \( u_y(t) \) is defined for \( t_i \leq t \leq c_i \) and \( u_y(c_i) \geq \tanh(\alpha - \arctanh(10^{-3})) \geq 0 \). \( \square \)

References


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