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# MATRIX SUMMABILITY OF UNBOUNDED SEQUENCES

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ABSTRACT. A well-known result of Mazur and Orlicz states that a matrix method strictly stronger than convergence sums not only bounded sequences but unbounded sequences. We consider the question of whether a matrix method strictly stronger than convergence will also sum a sequence with series terms (differences) constituting an unbounded sequence. This is equivalent to the series to sequence convergence domain of the matrix containing an unbounded sequence. A simple criterion is given showing in many cases the answer is positive. Counterexamples of three types are considered; triangles that are not perfect, perfect row finite matrices, and perfect triangles.

## 1. INTRODUCTION

It is well-known that a matrix method that is strictly stronger than convergence sums not only bounded sequences but unbounded sequences. We consider the question of whether a matrix method strictly stronger than convergence will also sum a sequence whose series terms (differences) constitute an unbounded sequence. We refer to such sequences briefly as series with unbounded terms. The answer is positive in many cases as is shown by Lemma 3.3. Counterexamples of three types are given in Propositions 4.1 and 4.2 and in Theorem 5.1 respectively. Propositions 4.1 and 4.2 deal with triangles that are not perfect (Einfolgenverfahren) and with perfect row-finite methods (generated by intertwining the rows of two matrices) respectively. A more delicate construction yields a suitable triangle that is also perfect (Theorem 5.1). In §6 we conclude with several remarks.

## 2. NOTATION AND TERMINOLOGY

Throughout we use notation and results given in the texts by Wilansky [11] and by Zeller-Beekmann [15]. Let $\omega$ denote the space of all sequences, $m$ all bounded sequences, $c$ the convergent sequences, $c_0$ sequences that converge to $0$, $cs = \{x\colon \sum_n x_n$ is convergent$\}$, $l_1 = \{x\colon \sum_n |x_n| < \infty\}$, and $\varphi$ all finitely nonzero sequences. If $A = (a_{nk})$ is an infinite matrix, the matrix method $A$ defines a sequence to sequence transformation, mapping the sequence $s$ (real

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or complex terms) to $t$

$$t_n = (As)_n = \sum_{k=0}^{\infty} a_{nk} s_k \,, \qquad n = 0, 1, 2, \ldots .$$

The convergence domain $c_A$ for the matrix method $A$ consists of those sequences $s$ for which $t = As$ exists and belongs to $c$. The $A$-limit is defined for $s \in c_A$ by $A\text{-}\lim s_k = \lim t_n$. The method $A$ is called *conservative* provided $c \subset c_A$ and *strictly stronger than convergence* if, moreover, $c \neq c_A$. A conservative matrix $A$ is regular provided $A\text{-}\lim s_k = \lim s_k$ for all $s \in c$ and is perfect if $c$ is dense in the FK space $c_A$, which is equivalent to the sequences $e = (1, 1, 1, \ldots)$, $e^0 = (1, 0, 0, \ldots)$, $e^1 = (0, 1, 0, \ldots)$, etc. forming a fundamental set, see for example [15]. The matrix $A$ is a triangle provided $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n$. Let $\Sigma$ denote the triangle of all ones, so that $A\Sigma$ is the series to sequence matrix method associated with $A$. Together with each $s$ we consider the corresponding $u$ given by the series terms (differences), i.e., $s_k = \sum_{i=0}^{k} u_i$ for all $k$. Thus we have $s = \sum u$ and $u = \sum^{-1} s$. We say that the method $A$ has SB if each $s \in c_A$ is bounded (i.e., $c_A \subseteq m$) and has UB, respectively UN, if for each $s \in c_A$, $u = \sum^{-1} s$ is bounded (i.e, $c_{A\Sigma} \subseteq m$), respectively $u \in c_0$ (i.e., $c_{A\Sigma} \subseteq c_0$). We remark here that UB and perfectness imply UN (by the fundamental set principle, see [15, Result 16 I]).

## 3. Criteria

The following known result is our starting point.

**Lemma 3.1.** *A matrix method $A$ that is strictly stronger than convergence violates SB (i.e., $c_A$ contains unbounded sequences).*

This result goes back to Mazur and Orlicz, see [15, Result 29 I].

The next result exhibits a relation between the properties SB and UB. The row-norms of the matrix method $A$ are given by $\sum_k |a_{nk}|$ for each $n \geq 0$.

**Lemma 3.2.** *A triangle $A$ has SB, respectively UB, if and only if the row-norms of $A^{-1}$, respectively $\Sigma^{-1}A^{-1}$, are bounded.*

The proof of the lemma is standard going back to Toeplitz. See for example [15, Result 32 I] or [11, Theorem 3.3, p. 5, Theorem 5, p. 12].

The final lemma of this section is applicable to many common methods.

**Lemma 3.3.** *If $\underline{\lim}_k \sup_n |a_{nk}| = 0$ then the matrix method $A$ violates SB and UB. If $A$ is a triangle and $\underline{\lim}_k |a_{kk}| = 0$ then $A$ violates SB and UB.*

The first part follows from [15, Result 25 I] (Spaltenmaximumktiterium), this criterion going back to Agnew [1]. For the second part apply Lemma 3.2, see [11, Theorem 7, p. 12].

## 4. Examples

The next two propositions exhibit two classes of matrix methods with property UB.

**Proposition 4.1.** *There is a regular matrix method $A$ strictly stronger than convergence having the following properties: The matrix is a triangle, the method is not perfect, and the method satisfies UB.*

*Proof.* Given any unbounded sequence $d$, there is a regular triangle $A$ such that $c_A$ consists of sequences of the form $\alpha d + y$ where $y \in c$, $\alpha \in \mathbb{C}$ ($c_A = c \oplus d$), see [15, Result 26 II] (Allgemeine Einfolgenverfahren). The result goes back to Mazur [8], Darevsky [3], and Zeller [14]. Here we choose any unbounded sequence $d$ with differences bounded. Then $c_{A\Sigma} = cs \oplus \sum^{-1} d \subseteq m$.

**Proposition 4.2.** *There is a regular matrix method $A$ strictly stronger than convergence having the following properties: The matrix is row-finite, the method is perfect, and the method satisfies UB.*

*Proof.* We begin with a row-finite matrix $B$ and $C = B\Sigma$. Thus $t = Bs = Cu$ (sequence and series forms respectively). Intertwining the rows of $C$ and $C + \mathrm{I}$ (I the identity matrix) we obtain a matrix $D$ and $A = D\Sigma^{-1}$ such that $c_A$ consists of the $s \in c_B$ with $u \in c_0$. That is, the series domain $c_D$ equals $c_C \cap c_0$. As is well known $c_C$ and $c_0$ are BK-spaces with norms $\sup_n |t_n|$ and $\sup_m |u_m|$ respectively. The space $c_D$ is then a BK-space with the sum of these two norms. Now let $B$ be the Cesàro matrix of order one. Then $A$ is regular and strictly stronger than convergence (there are divergent Cesàro summable series that are in $c_0$). Finally we show that $A$ is perfect. The BK-space $c_C$ has $C$-sectional convergence. i.e.,

$$\lim_n \|c_{n0}u_0e^0 + c_{n1}u_1e^1 + \cdots + c_{nn}u_ne^n - u\|_{c_C} = 0$$

for each Cesàro summable series $u$ (by Hardy-Bohr, see [14, 7]). Since the BK-space $c_0$ has AK (Abschnittskonvergenz), i.e., $\{e^n\}_{n\geq 0}$ is a basis, it follows that $c_0$ also has $C$-sectional convergence. Together this yields $C$-sectional convergence in the BK-space $c_D$. In particular, $c_D$ will then have AD (Abschnittsdiche), i.e., $\varphi$ is dense, and it follows that the matrix $A$ is perfect.

## 5. Main result

We now combine the properties "triangle" and "perfect" that appear separately in the examples of §4. The construction here is more delicate and allows for many modifications.

**Theorem 5.1.** *There is a regular matrix method $A$ strictly stronger than convergence having the following properties: The matrix is a triangle, the method is perfect, and the method satisfies UN.*

*Proof.* We use the matrix $Z = Z_{1/2}$ (Zweierverfahren), see [15, p. 125], spread it and insert suitable rows. Our precise definition of the triangle $A$ begins with $a_{00} = a_{10} = a_{11} = \frac{1}{2}$. Then we consider a fixed $j \geq 1$ and define $p = (j-1)^2$, $q = j^2$, $r = (j+1)^2$, $\delta = 1/4j$, and

$$t_n = \alpha_n s_p + \beta_n s_q + \frac{1}{2} s_n \quad \text{for } q < n \leq r,$$

where

$$\alpha_n + \beta_n = \frac{1}{2}, \qquad \alpha_{q+1} = \frac{1}{2}, \quad \alpha_{q+2} = \frac{1}{2} - \delta, \quad \alpha_{q+3} = \frac{1}{2} - 2\delta, \ldots, \alpha_r = 0.$$

For $n = r$ we have a row as in $Z$, with the terms spread. We consider

$$t_n - t_{n-1} = (\alpha_n - \alpha_{n-1})s_p + (\beta_n - \beta_{n-1})s_q + \frac{1}{2}(s_n - s_{n-1})$$

$(u_n = s_n - s_{n-1})$ where in this relation we let $\alpha_p = \frac{1}{2}$ and $\beta_q = 0$. For $s \in c_A$ the first two terms on the right-hand side are $o(1)$ by the definition of the coefficients and since $s_{j^2} = o(j)$ (use the order condition for $Z$). Thus $u \in c_0$. Further $A$ is strictly stronger than convergence (take e.g., $s_{j^2} = (-1)^j$ and define the remaining $s_k$ by $t_k = 0$).

Finally to show that $A$ is perfect we use [15, Result 23 I] (or see [11, Theorem 4, p. 42]). That is, we show that $A$ is of type $M$. Thus suppose $g \in l_1$ and $gA = 0$. It is immediate that $g_n = 0$ for $n \neq j^2$, $j \geq 0$. Moreover, it follows that for $n = j^2$, $j \geq 0$, $g_n = (-1)^j g_0$, and hence the terms must vanish for any $g \in l_1$. Thus we see that $A$ is perfect.

## 6. Remarks

The Nörlund polynomial methods provide a collection of matrix methods that, when strictly stronger than convergence, are easily seen to violate UB.

For a polynomial $p(z) = \sum_{n=0}^{N} p_n z^n$ of degree $N$ the Nörlund polynomial method $N_p$ is given by $N_p[n, k] = p_{n-k}$ for $k \leq n$ and 0 otherwise where $p_n = 0$ for $n > N$. If $p(z) = \prod_{i=1}^{N}(a_i + b_i z)$, since Nörlund polynomial methods commute, then for each $i$, $1 \leq i \leq N$, $c_{N_{q_i}} \subseteq c_{N_p}$ where $q_i(z) = a_i + b_i z$. Without loss of generality we may consider only polynomial methods generated by polynomials of the form $p(z) = a + bz$. We may also assume $|a/b| \leq 1$, for otherwise the method will be equivalent to convergence, see [5, Theorem 4.6]. Moreover, $N_p$ is of type $M$, and hence perfect, see [11, Theorem 4, p. 42], if and only if $|a/b| \geq 1$, see [4]. If $N_p$ is generated by $p(z) = a + bz$ and is strictly stronger than convergence by Lemma 3.2, $N_p$ violates UB and consequently sums a series with unbounded terms, see also [6]. Hence by the previous remarks Nörlund polynomial methods strictly stronger than convergence violate UB. This can be extended to include a class of little Nörlund means. That is, let $p(z) = \sum_{n=0}^{\infty} p_n z^n$; then the little Nörlund mean $N_p$ associated with this series is given by $N_p[n, k] = p_{n-k}$ for $k \leq n$ and 0 otherwise. If the radius of convergence of the power series is greater than one and $p(\alpha) = 0$, then $p(z) = (z - \alpha)q(z)$ where $q(z) = \sum_{k=0}^{\infty} q_k z^k$, with the radius of convergence of $q(z)$ also greater than one. Moreover, $q \in l_1$ and $N_p = N_q N_r$ where $r(z) = (z - \alpha)$. If $N_p$ is stronger than convergence then there must exist some $\alpha$ with $|\alpha| \leq 1$, and the result now follows.

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