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-REPRESENTATIONS OF THE TRACE-CLASS OF AN H-ALGEBRA

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ABSTRACT. The aim of this note is to characterize the cyclic and the irreducible *-representations of the trace-class of a proper H*-algebra.

Throughout this paper A denotes a proper H*-algebra (i.e., A is a Banach *-algebra whose norm is a Hilbert space norm such that $\langle x, yz^* \rangle = \langle xz, y \rangle = \langle z, x^*y \rangle$ for every $x, y, z \in A$). A projection in A is a nonzero element e of A such that $e^2 = e = e^*$; e is called primitive if it cannot be represented as a sum of two mutually orthogonal primitive projections in A. A maximal family of mutually orthogonal primitive projections is called projection base. An element $a \in A$ is said to be positive ($a \geq 0$) if $\langle ax, x \rangle \geq 0$ for every $x \in A$. For each $a \in A$ there exists a unique positive element $|a|$ of A such that $|a|^2 = a^*a$.

By the trace-class of A we mean the set $\tau(A) = \{xy : x, y \in A\}$ that is dense in A. If $a \in A$, then the following assertions are equivalent:

(i) $a \in \tau(A)$.
(ii) $|a| \in \tau(A)$.
(iii) There exists a positive element b of A such that $b^2 = |a|$.
(iv) $\sum_\alpha \langle |a| e_\alpha, e_\alpha \rangle < \infty$ for some projection base $\{e_\alpha\}$ in A.

There is a positive linear functional tr (called trace) on $\tau(A)$ such that $\operatorname{tr} xy^* = \operatorname{tr} y^*x = \langle x, y \rangle$ and $\overline{\operatorname{tr} a} = \operatorname{tr} a^*$ for every $x, y \in A$ and $a \in \tau(A)$. One can define a Banach algebra norm τ on $\tau(A)$ by the formula $\tau(a) = \operatorname{tr}|a|$ ($a \in \tau(A)$). Denote by $R(A)$ the set of right centralizers on A, i.e., let

$$R(A) = \{S \in B(A) : \ S(xy) = (Sx)y \ (\forall x, y \in A)\},$$

where $B(A)$ denotes the set of bounded linear operators on A. It is trivial that L_x, the operator of the left multiplication by x, is in $R(A)$ for every $x \in A$. $R(A)$ is isomorphic and isometric to $\tau(A)^*$.

As for the detailed discussion of proper H*-algebras and their trace-classes as well as the proofs of the above statements we refer to [1, 5, 6].

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A positive linear functional f on a Banach *-algebra B is called representable if there is a Hilbert space H and a *- representation $x \mapsto T_x$ of B on H with cyclic vector $b \in H$ such that $f(x) = \langle T_x b, b\rangle$ $(x \in B)$. In [2, Theorem 37.11] it was stated that a positive linear functional $f\colon B \to \mathbb{C}$ is representable if and only if there exists a positive constant $c \in \mathbb{R}$ for which

$$|f(x)|^2 \le c\, f(x^*x) \qquad (x \in B).$$

Unfortunately, the proof presented there is incomplete since it uses the hermicity of the functional. For a correct proof see [4].

We begin with the following two basic lemmas.

Lemma 1. *Let $S \in R(A)$. Then the following assertions are equivalent*:

(i) $\sum_\alpha \langle |S| e_\alpha, e_\alpha\rangle < \infty$ *for some projection base* $\{e_\alpha\}$ *in* A.
(ii) *There exists a unique* $a \in \tau(A)$ *such that* $S = L_a$.

Proof. Suppose that (i) holds. From the inequality $S^*S \le \|S\||S|$ we have $\sum_\alpha \|Se_\alpha\|^2 < \infty$. Since S is a right centralizer, one can easily verify that $\{Se_\alpha\}$ is a mutually orthogonal vector system. Let $a = \sum_\alpha Se_\alpha$. Then

$$L_a x = ax = (\sum\nolimits_\alpha Se_\alpha)x = S(\sum\nolimits_\alpha e_\alpha x) = Sx \qquad (x \in A),$$

where we have used the fact that $x = \sum_\alpha e_\alpha x$ for every $x \in A$. Now $L_{|a|} = |L_a| = |S|$ implies that $a \in \tau(A)$. The uniqueness of a is obvious.

The other implication is easy to prove.

Lemma 2. *Let $a \in \tau(A)$ be positive. Then*

$$\tau(a) = \inf\{c \in \mathbb{R} : c \ge 0,\ \ |\operatorname{tr} ax|^2 \le c \operatorname{tr} ax^*x \ \ (x \in \tau(A))\}.$$

Proof. Consider the semi-inner product B on $\tau(A)$ defined by

$$B(x, y) = \langle ax, y\rangle \qquad (x, y \in \tau(A)).$$

The Cauchy-Schwarz inequality implies that

$$|\langle axe, e\rangle|^2 = |\langle ax, e\rangle|^2 \le \langle ae, e\rangle\langle ax, x\rangle \qquad (x \in \tau(A)),$$

where e is an arbitrary projection in A. Now it follows that

$$|\operatorname{tr} ax^*|^2 = |\operatorname{tr}(ax^*)^*|^2 = |\operatorname{tr} ax|^2 \le \tau(a) \operatorname{tr} x^*ax = \tau(a) \operatorname{tr} axx^* \qquad (x \in \tau(A)).$$

If $c \in \mathbb{R}$, $c \ge 0$ such that $|\operatorname{tr} ax|^2 \le c \operatorname{tr} ax^*x$ $(x \in \tau(A))$, then for every projection e in A we have $\langle ae, e\rangle = \operatorname{tr} ae \le c$, which implies that $\tau(a) \le c$.

Our first theorem characterizes the representable positive linear functionals on $\tau(A)$.

Theorem 1. *Let $f\colon \tau(A) \to \mathbb{C}$ be a positive linear functional. Then the following assertions are equivalent*:

(i) f *is representable.*
(ii) *There exists a unique positive element* a *of* $\tau(A)$ *such that* $f(x) = \operatorname{tr} ax$ *for every* $x \in \tau(A)$.
(iii) *There exists a unique positive element* b *of* A *such that* $f(x) = \langle L_x b, b\rangle$ *for every* $x \in \tau(A)$.

Proof. Let f be representable. Then there is a positive constant $c \in \mathbb{R}$ such that $|f(x)|^2 \leq c\, f(x^*x)$ $(x \in \tau(A))$. Since $f \in \tau(A)^*$, by [5, Theorem 2], there is a positive operator $S \in R(A)$ for which $f(x) = \operatorname{tr} Sx$ $(x \in \tau(A))$. If $e \in A$ is a projection, then we have $|\operatorname{tr} Se|^2 \leq c \operatorname{tr} Se$, i.e., $\operatorname{tr} Se \leq c$. Since it holds for every projection in A, we can conclude that $\sum_\alpha \langle Se_\alpha, e_\alpha \rangle \leq c$ for every projection base $\{e_\alpha\}$ in A. By Lemma 1 there is a positive element a in $\tau(A)$ such that $S = L_a$. The uniqueness of a follows from the density of $\tau(A)$ in A.

To the implication (ii) $\Rightarrow$ (iii), let $b \in A$ be the positive square root of a. The remainder part of the statement is easy to check.

As a consequence of the above theorem and [3, Lemma (4.5.8)] we have the following

Theorem 2. *Let $b \in A$. If H_b denotes the closure of the subspace $\tau(A)b$ in A then $x \mapsto L_x \restriction H_b$ is a *-representation of $\tau(A)$ with cyclic vector b. Moreover, every cyclic *-representation of $\tau(A)$ is unitarily equivalent to a representation of this kind.*

Proof. The only thing that has to be proved is $b \in H_b$ for every $b \in A$. But it follows from the fact that the projections in A belong to $\tau(A)$.

In what follows, let

$$P = \{f \in \tau(A)^* : f \text{ is positive and } |f(x)|^2 \leq f(x^*x) \ \ (x \in \tau(A))\}.$$

By Theorem 1, for every representable positive linear functional f on $\tau(A)$ there exists a unique positive member a of $\tau(A)$ such that $f(x) = \operatorname{tr} ax$ $(x \in \tau(A))$. Now, by Lemma 2, $f \in P$ if and only if $\tau(a) \leq 1$. If f is not identically zero, then by [3, Corollary (4.6.5)], one can easily verify that f is an extremal point of P if and only if the conditions $\bar{a} \in \tau(A)$, $\bar{a} \geq 0$, $\tau(\bar{a}) \leq 1$, and $\lambda a - \bar{a} \geq 0$ for some $0 < \lambda \in \mathbb{R}$ imply that there is a $0 \leq \mu \in \mathbb{R}$ such that $\mu a = \bar{a}$.

Theorem 3. *Let $0 \neq f \in P$ and a be the unique element of $\tau(A)$ corresponding f as above. Then f is an extremal point of P if and only if there exists a primitive projection e in A for which $a = e/\|e\|^2$.*

Proof. Necessity. Suppose that f is an extremal point of P. It is easy to see that $\tau(a) = 1$. Let $a = \sum_n \lambda_n e_n$ be the spectral representation of a where $0 < \lambda_n \in \mathbb{R}$ and $\{e_n\}$ is a sequence of mutually orthogonal primitive projections (see [6, Corollary 1]). Let $\bar{a} = e_1/\|e_1\|^2$. Then $\bar{a} \in \tau(A)$, $\bar{a} \geq 0$, and $\tau(\bar{a}) = 1$. Moreover, for $\lambda = 1/\lambda_1\|e_1\|^2$ we have $\lambda a - \bar{a} \geq 0$. Consequently, there exists an $0 \leq \mu \in \mathbb{R}$ such that $\mu a = e_1/\|e_1\|^2$. Taking traces we arrive at

$$\mu = \mu \operatorname{tr} a = (1/\|e_1\|^2) \operatorname{tr} e_1 = 1.$$

Suficiency. Let $a = e/\|e\|^2$ where e is a primitive projection in A. Suppose that $\bar{a} \in \tau(A)$, $0 \neq \bar{a} \geq 0$ such that $\tau(\bar{a}) \leq 1$ and $\lambda a - \bar{a} \geq 0$ for some $0 < \lambda \in \mathbb{R}$. Let $\bar{a} = \sum_n \lambda_n e_n$ be the spectral representation of $\bar{a}$. Then, for every fixed n, we have $\lambda e/\|e\|^2 \geq \lambda_n e_n$. If we extend the singleton $\{e\}$ to a projection base, then the first structure theorem of proper H*-algebras (c.f. [1, Theorem 4.1]) implies that $e_n A \subset eA$. Since eA is a minimal closed right ideal thus $e_n A = eA$. It is known that the projection of $x \in A$ on the closed

right ideal $\bar{e}A$, where $\bar{e}$ is an arbitrary projection in A, is $\bar{e}x$. Consequently, we have $e_n = ee_n = e_n e = e$, which implies that there is a $0 < \mu \in \mathbb{R}$ for which $\mu a = \bar{a}$. This completes the proof.

Using the notation of Theorem 2, by [3, Theorem (4.6.4)], it is easy to prove our final result, which follows.

Theorem 4. *Let e be a primitive projection in A. Then $x \mapsto L_x \restriction H_e$ is a nonzero irreducible $*$- representation of $\tau(A)$. Moreover, every irreducible $*$-representation of $\tau(A)$ is unitarily equivalent to a representation of this kind.*

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