*-REPRESENTATIONS OF THE TRACE-CLASS OF AN H*-ALGEBRA

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Abstract. The aim of this note is to characterize the cyclic and the irreducible *-representations of the trace-class of a proper H*-algebra.

Throughout this paper $A$ denotes a proper H*-algebra (i.e., $A$ is a Banach *-algebra whose norm is a Hilbert space norm such that $\langle x, yz^* \rangle = \langle xz, y \rangle = \langle z, x^*y \rangle$ for every $x, y, z \in A$). A projection in $A$ is a nonzero element $e$ of $A$ such that $e^2 = e = e^*$; $e$ is called primitive if it cannot be represented as a sum of two mutually orthogonal primitive projections in $A$. A maximal family of mutually orthogonal primitive projections is called projection base.

An element $a \in A$ is said to be positive ($a \geq 0$) if $\langle ax, x \rangle \geq 0$ for every $x \in A$. For each $a \in A$ there exists a unique positive element $|a|$ of $A$ such that $|a|^2 = a^*a$.

By the trace-class of $A$ we mean the set $\tau(A) = \{xy : x, y \in A\}$ that is dense in $A$. If $a \in A$, then the following assertions are equivalent:

(i) $a \in \tau(A)$.
(ii) $|a| \in \tau(A)$.
(iii) There exists a positive element $b$ of $A$ such that $b^2 = |a|$.
(iv) $\sum_\alpha \langle |a| e_\alpha, e_\alpha \rangle < \infty$ for some projection base $\{e_\alpha\}$ in $A$.

There is a positive linear functional $\text{tr}$ (called trace) on $\tau(A)$ such that $\text{tr}xy^* = \text{tr}y^*x = \langle x, y \rangle$ and $\text{tr}a = \text{tr}a^*$ for every $x, y \in A$ and $a \in \tau(A)$. One can define a Banach algebra norm $\tau$ on $\tau(A)$ by the formula $\tau(a) = \text{tr}|a|$ ($a \in \tau(A)$). Denote by $R(A)$ the set of right centralizers on $A$, i.e., let

$R(A) = \{S \in B(A) : S(xy) = Sx \overline{y} \quad (\forall x, y \in A)\}$,

where $B(A)$ denotes the set of bounded linear operators on $A$. It is trivial that $L_x$, the operator of the left multiplication by $x$, is in $R(A)$ for every $x \in A$. $R(A)$ is isomorphic and isometric to $\tau(A)^*$.

As for the detailed discussion of proper H*-algebras and their trace-classes as well as the proofs of the above statements we refer to [1, 5, 6].

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A positive linear functional $f$ on a Banach $*$-algebra $B$ is called representable if there is a Hilbert space $H$ and a $*$-representation $x \mapsto T_x$ of $B$ on $H$ with cyclic vector $b \in H$ such that $f(x) = \langle T_x b, b \rangle$ ($x \in B$). In [2, Theorem 37.11] it was stated that a positive linear functional $f : B \to \mathbb{C}$ is representable if and only if there exists a positive constant $c \in \mathbb{R}$ for which

$$|f(x)|^2 \leq c f(x^*x) \quad (x \in B).$$

Unfortunately, the proof presented there is incomplete since it uses the hermiticity of the functional. For a correct proof see [4].

We begin with the following two basic lemmas.

**Lemma 1.** Let $S \in \mathcal{R}(A)$. Then the following assertions are equivalent:

(i) $\sum_a \langle S e_a, e_a \rangle < \infty$ for some projection base $\{e_a\}$ in $A$.

(ii) There exists a unique $a \in \tau(A)$ such that $S = L_a$.

**Proof.** Suppose that (i) holds. From the inequality $S^*S \leq \|S\|\|S\|$ we have $\sum_a \|S e_a\|^2 < \infty$. Since $S$ is a right centralizer, one can easily verify that $\{S e_a\}$ is a mutually orthogonal vector system. Let $a = \sum_a S e_a$. Then

$$L_a x = ax = (\sum_a S e_a) x = S (\sum_a e_a x) = S x \quad (x \in A),$$

where we have used the fact that $x = \sum_a e_a x$ for every $x \in A$. Now $L_{|a|} = |L_a| = |S|$ implies that $a \in \tau(A)$. The uniqueness of $a$ is obvious.

The other implication is easy to prove.

**Lemma 2.** Let $a \in \tau(A)$ be positive. Then

$$\tau(a) = \inf \{ c \in \mathbb{R} : c > 0, \ |\text{tr} ax|^2 \leq c \text{tr} ax^*x \ (x \in \tau(A)) \}.$$

**Proof.** Consider the semi-inner product $B$ on $\tau(A)$ defined by

$$B(x, y) = \langle ax, y \rangle \quad (x, y \in \tau(A)).$$

The Cauchy-Schwarz inequality implies that

$$\|\langle axe, e \rangle\|^2 = \|\langle ax, e \rangle\|^2 \leq \langle ae, e \rangle \langle ax, x \rangle \quad (x \in \tau(A)),$$

where $e$ is an arbitrary projection in $A$. Now it follows that

$$|\text{tr} ax^*|^2 = |\text{tr}(ax^*)^*|^2 = |\text{tr} ax|^2 \leq \tau(a) \text{tr} x^*ax = \tau(a) \text{tr} axx^* \quad (x \in \tau(A)).$$

If $c \in \mathbb{R}$, $c \geq 0$ such that $|\text{tr} ax|^2 \leq c \text{tr} ax^*x$ ($x \in \tau(A)$), then for every projection $e$ in $A$ we have $\langle ae, e \rangle = \text{tr} ae \leq c$, which implies that $\tau(a) \leq c$.

Our first theorem characterizes the representable positive linear functionals on $\tau(A)$.

**Theorem 1.** Let $f : \tau(A) \to \mathbb{C}$ be a positive linear functional. Then the following assertions are equivalent:

(i) $f$ is representable.

(ii) There exists a unique positive element $a$ of $\tau(A)$ such that $f(x) = \text{tr} ax$ for every $x \in \tau(A)$.

(iii) There exists a unique positive element $b$ of $A$ such that $f(x) = \langle L_x b, b \rangle$ for every $x \in \tau(A)$. 

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Proof. Let $f$ be representable. Then there is a positive constant $c \in \mathbb{R}$ such that $|f(x)|^2 \leq c f(x^*x)$ $(x \in \tau(A))$. Since $f \in \tau(A)^*$, by [5, Theorem 2], there is a positive operator $S \in \mathcal{R}(A)$ for which $f(x) = \text{tr} S x$ $(x \in \tau(A))$. If $e \in A$ is a projection, then we have $|\text{tr} S e|^2 \leq c \text{tr} S e$, i.e., $\text{tr} S e \leq c$. Since it holds for every projection in $A$, we can conclude that $\sum_a \langle S e_a, e_a \rangle \leq c$ for every projection base $\{e_a\}$ in $A$. By Lemma 1 there is a positive element $a$ in $\tau(A)$ such that $S = L_a$. The uniqueness of $a$ follows from the density of $\tau(A)$ in $A$.

To the implication $(ii) \Rightarrow (iii)$, let $b \in A$ be the positive square root of $a$. The remainder part of the statement is easy to check.

As a consequence of the above theorem and [3, Lemma (4.5.8)] we have the following

**Theorem 2.** Let $b \in A$. If $H_b$ denotes the closure of the subspace $\tau(A)b$ in $A$ then $x \mapsto L_x | H_b$ is a $*$-representation of $\tau(A)$ with cyclic vector $b$. Moreover, every cyclic $*$-representation of $\tau(A)$ is unitarily equivalent to a representation of this kind.

**Proof.** The only thing that has to be proved is $b \in H_b$ for every $b \in A$. But it follows from the fact that the projections in $A$ belong to $\tau(A)$.

In what follows, let

$$ P = \{ f \in \tau(A)^* : f \text{ is positive and } |f(x)|^2 \leq f(x^*x) \ (x \in \tau(A)) \}. $$

By Theorem 1, for every representable positive linear functional $f$ on $\tau(A)$ there exists a unique positive member $a$ of $\tau(A)$ such that $f(x) = \text{tr} a x$ $(x \in \tau(A))$. Now, by Lemma 2, $f \in P$ if and only if $\tau(a) \leq 1$. If $f'$ is not identically zero, then by [3, Corollary (4.6.5)], one can easily verify that $f'$ is an extremal point of $P$ if and only if the conditions $a \in \tau(A)$, $a \geq 0$, $\tau(a) \leq 1$, and $\lambda a - a \geq 0$ for some $0 < \lambda \in \mathbb{R}$ imply that there is a $0 \leq \mu \in \mathbb{R}$ such that $\mu a = a$.

**Theorem 3.** Let $0 \neq f \in P$ and $a$ be the unique element of $\tau(A)$ corresponding $f$ as above. Then $f$ is an extremal point of $P$ if and only if there exists a primitive projection $e$ in $A$ for which $a = e/\|e\|^2$.

**Proof.** Necessity. Suppose that $f$ is an extremal point of $P$. It is easy to see that $\tau(a) = 1$. Let $a = \sum_n \lambda_n e_n$ be the spectral representation of $a$ where $0 < \lambda_n \in \mathbb{R}$ and $\{e_n\}$ is a sequence of mutually orthogonal primitive projections (see [6, Corollary 1]). Let $\bar{a} = e_1/\|e_1\|^2$. Then $\bar{a} \in \tau(A)$, $\bar{a} \geq 0$, and $\tau(\bar{a}) = 1$. Moreover, for $\lambda = 1/\|e_1\|^2$ we have $\lambda a - \bar{a} \geq 0$. Consequently, there exists an $0 \leq \mu \in \mathbb{R}$ such that $\mu a = e_1/\|e_1\|^2$. Taking traces we arrive at

$$ \mu = \mu \text{tr} e = (1/\|e_1\|^2) \text{tr} e_1 = 1. $$

Sufficiency. Let $a = e/\|e\|^2$ where $e$ is a primitive projection in $A$. Suppose that $a \in \tau(A)$, $0 \neq a \geq 0$ such that $\tau(a) \leq 1$ and $\lambda a - \bar{a} \geq 0$ for some $0 < \lambda \in \mathbb{R}$. Let $\bar{a} = \sum_n \lambda_n e_n$ be the spectral representation of $\bar{a}$. Then, for every fixed $n$, we have $\lambda e/\|e\|^2 \geq \lambda_n e_n$. If we extend the singleton $\{e\}$ to a projection base, then the first structure theorem of proper $H^*$-algebras (c.f. [1, Theorem 4.1]) implies that $e_n A \subset e A$. Since $e A$ is a minimal closed right ideal thus $e_n A = e A$. It is known that the projection of $x \in A$ on the closed
right ideal \( eA \), where \( e \) is an arbitrary projection in \( A \), is \( ex \). Consequently, we have \( e_n = ee_n = e_ne = e \), which implies that there is a \( 0 < \mu \in \mathbb{R} \) for which \( \mu a = a \). This completes the proof.

Using the notation of Theorem 2, by [3, Theorem (4.6.4)], it is easy to prove our final result, which follows.

**Theorem 4.** Let \( e \) be a primitive projection in \( A \). Then \( x \mapsto L_x \upharpoonright H_e \) is a nonzero irreducible *-representation of \( \tau(A) \). Moreover, every irreducible *-representation of \( \tau(A) \) is unitarily equivalent to a representation of this kind.

**References**