CHARACTERIZATION OF SEPARABLE METRIC R-TREES

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Abstract. An R-tree (X, d) is a uniquely arcwise connected metric space in which each arc is isometric to a subarc of the reals. R-trees arise naturally in the study of groups of isometries of hyperbolic space. Two of the authors had previously characterized R-trees topologically among metric spaces. The purpose of this paper is to provide a simpler proof of this characterization for separable metric spaces. The main theorem is the following: Let (X, r) be a separable metric space. Then the following are equivalent:

1. X admits an equivalent metric d such that (X, d) is an R-tree.
2. X is locally arcwise connected and uniquely arcwise connected.

The method of proving that (2) implies (1) is to "improve" the metric r through a sequence of equivalent metrics of which the first is monotone on arcs, the second is strictly monotone on arcs, and the last is convex, as desired.

1. Introduction

1.1. R-trees. An R-tree (X, d) is a uniquely arcwise connected metric space (see definition below) in which each arc is isometric to a subarc of the reals. ("Uniquely" is superfluous in this definition.) Actions on R-trees can be seen as ideal points in the compactification of groups of isometries of hyperbolic space [Mr, Be, MrS]. The R-trees considered in these applications are always separable. In this paper we provide a much simpler proof of the characterization of R-trees among separable metric spaces than the proof provided for metric spaces in general in [MO].

1.2. Main Theorem (Characterization Theorem). (See Theorem 5.1 of [MO].) Let (X, r) be a separable metric space. Then the following are equivalent:

1. X admits an equivalent metric d such that (X, d) is an R-tree.
2. X is locally arcwise connected and uniquely arcwise connected.

If r is a metric on a space X, by Br(x, ε) we denote the open ball about x ∈ X of radius ε > 0 in the metric r. That (1) implies (2) follows from the...
observation that in an $R$-tree $(X, d)$, if $d(x, z) = \delta$, then there is an arc from $x$ to $z$, at each point $y \neq z$ of which $d(x, y) < \delta$. (See the definition of a convex metric and Proposition 1.4 below.) Hence, for all $\varepsilon > 0$, $B_d(x, \varepsilon)$ is arcwise connected.

The proof that (2) implies (1) proceeds through a series of lemmas, which we present in §2. These lemmas gradually “improve” the metric $r$ on $X$ through a sequence of equivalent metrics $\rho$, $\rho^*$, and $d$, such that $\rho$ is monotone on arcs, $\rho^*$ is strictly monotone on arcs, and, finally, $d$ is convex, so that $(X, d)$ is an $R$-tree.

1.3. Definitions. An arc $A$ in a space $X$ is the image $A = \varepsilon([0, 1])$ of an embedding $\varepsilon: [0, 1] \rightarrow X$; the end points of $A$ are $\varepsilon(0)$ and $\varepsilon(1)$. We say a space $X$ is (uniquely) arcwise connected iff given $x \neq y \in X$, there is a (unique) arc $A \subset X$ whose end points are $x$ and $y$. In a uniquely arcwise connected space, the intersection of any two arcwise connected subsets is again arcwise connected.

Let $X$ be a uniquely arcwise connected space with $x \neq y \in X$. By $[x, y]$ we denote the unique arc in $X$ whose end points are $x$ and $y$; $[x, x]$ denotes $\{x\}$. Let $p \in X$ be fixed. Since $X$ is uniquely arcwise connected, we have for all $x, y \in X$, $[p, x] \cap [p, y] = [p, z]$ for some $z \in X$ (possibly, $[p, z]$ is degenerate). Define a meet function $\wedge_p: X \times X \rightarrow X$ with respect to $p$ by setting $x \wedge_p y = z$, where $z$ is defined as above. We will drop the $p$ subscript where this can be done without confusion.

We say a space $X$ is locally arcwise connected iff for every point $x \in X$ and every open neighborhood $U$ of $x$, there is an arcwise connected open neighborhood $V$ of $x$ with $V \subset U$.

Let $(X, r)$ be a uniquely arcwise connected metric space. We say that $r$ is monotone on arcs (respectively, strictly monotone on arcs) iff for all $x \neq z \in X$, for every $y \in [x, z)$, $r(x, y) \leq r(x, z)$ (respectively, $r(x, y) < r(x, z)$). We call the metric $r$ on $X$ a convex metric iff for all $x, z \in X$, for all $y \in [x, z]$, $r(x, z) = r(x, y) + r(y, z)$. A convex metric on $X$ is strictly monotone on arcs, though not necessarily conversely. The following equivalence is easy to prove:

1.4. Proposition. Let $(X, d)$ be a uniquely arcwise connected metric space. Then the following are equivalent:

- $(X, d)$ is an $R$-tree.
- $d$ is a convex metric on $X$.

2. Improving the metric

In this section we assume that $(X, r)$ is a uniquely arcwise connected, locally arcwise connected, separable metric space. We may assume without loss of generality that $r$ is bounded. In the following lemmas we show how to improve the metric $r$ on $X$, through a series of equivalent metrics, to a metric $d$ on $X$ with respect to which $X$ is an $R$-tree.

2.1. Lemma. The metric $\rho$ on $X$, defined by $\rho(x, y) = \text{diam}_r([x, y])$, is bounded, equivalent to $r$, and monotone on arcs.

Proof. If $y \in [x, z]$, then $[x, y] \subset [x, z]$, which implies $\text{diam}_r([x, y]) \leq \text{diam}_r([x, z])$; so $\rho$ is monotone on arcs and bounded by the bound of $r$. 
That $p$ is a metric is easy to verify. We verify the triangle inequality: since $X$ is uniquely arcwise connected and $[x, y] \cap [y, z] \neq \emptyset$, we have $[x, z] \subset [x, y] \cup [y, z]$. Thus,
\[
\text{diam}_r([x, z]) \leq \text{diam}_r([x, y] \cup [y, z]) \leq \text{diam}_r([x, y]) + \text{diam}_r([y, z]).
\]
Hence, $p(x, z) \leq p(x, y) + p(y, z)$.

Since $p(x, y) = \text{diam}_r([x, y]) \geq r(x, y)$, we have $B_p(x, \varepsilon) \subset B_r(x, \varepsilon)$; so the topology on $X$ generated by $p$ is finer than the topology on $X$ generated by $r$. Conversely, to see that $r$ is finer than $p$, let $\varepsilon > 0$ and $x \in X$ be given. Since $X$ is locally arcwise connected, there is an arcwise connected $r$-open neighborhood $U$ of $x$ such that $U \subset B_r(x, \varepsilon/2)$. Then for all $y \in U$, $r(x, y) = \text{diam}_r([x, y]) < \varepsilon$. Therefore, $U \subset B_p(x, \varepsilon)$.

2.2. **Lemma.** Let $D$ be a countable dense subset of $X$, and let $\{J_i\}_{i=1}^{\infty}$ be the collection of all arcs between different points of $D$. For each $i$, let $d_i$ be a convex metric on $J_i$ such that $\text{diam}_{d_i}(J_i) = 2^{-i}$. Then the metric $p^*$ on $X$, defined by
\[
p^*(x, y) = p(x, y) + \sum_{i=1}^{\infty} \text{diam}_{d_i}(J_i \cap [x, y]),
\]
is bounded, equivalent to $p$, and strictly monotone on arcs.

**Proof.** From the fact that $X$ is uniquely arcwise connected, it is easy to show that the following hold for all $x, y, z \in X$, for all $i$:

1. $J_i \cap [x, y]$ is either empty, a point, or an arc,
2. $[x, z] \subset [x, y] \cup [y, z]$, and
3. $J_i \cap ([x, y] \cup [y, z])$ is either empty, a point, or an arc.

Consequently, we have
\[
J_i \cap [x, z] \subset J_i \cap ([x, y] \cup [y, z]) = (J_i \cap [x, y]) \cup (J_i \cap [y, z])
\]
and thus,
\[
\text{diam}_{d_i}(J_i \cap [x, z]) \leq \text{diam}_{d_i}(J_i \cap [x, y]) + \text{diam}_{d_i}(J_i \cap [y, z]).
\]
The triangle inequality now follows from the definition of $p^*$ by considering each term in the sum. The remaining properties of a metric are easy to check. The bound on $p^*$ is the bound on $p$ plus 1.

Since the set of end points of the $J_i$'s is dense in $X$ and $X$ is locally arcwise connected and uniquely arcwise connected, it follows that if $U$ and $V$ are disjoint closed neighborhoods of $x$ and $y$, respectively, then there is some index $i$ such that for $J_i = [a, b]$, we have $a, x \wedge y a \in U$ and $b, y \wedge x b \in V$; so $J_i \cap [x, y] \supset [x, y] - (U \cup V)$. In particular, $J_i \cap [x, y]$ is nondegenerate. (It may be helpful to draw a figure.) It now follows from the definition of $p^*$ that for all $x \neq y \in X$, $p^*(x, y) > p(x, y)$. Therefore, $p^*$ is finer than $p$.

Now suppose $y \in [x, z]$. It follows that for all $i$,
\[
diam_{d_i}(J_i \cap [x, y]) \leq \text{diam}_{d_i}(J_i \cap [x, z]).
\]
But by the argument in the preceding paragraph (choose $U$ about $x$ and $V$ about $z$ to both miss $y$), there is an index $i$ such that strict inequality obtains in $(*)$, since $d_i$ is convex, hence strictly monotone, on $J_i$. It follows that $p^*(x, y) < p^*(x, z)$; so, $p^*$ is strictly monotone on arcs.
To see that \( \rho \) is finer than \( \rho^* \), let \( \varepsilon > 0 \) and \( x \in X \) be given. Choose \( N \) such that \( \sum_{i=N+1}^{\infty} 2^{-i} < \varepsilon/2 \). For \( i = 1, 2, \ldots, N \), partition each \( J_i \) into subarcs of \( d_i \)-diameter at most \( \varepsilon/8N \). Since \( X \) is locally arcwise connected, we may choose an arcwise connected \( \rho \)-open neighborhood \( U \) of \( x \) such that \( U \subset B_\rho(x, \varepsilon/4) \) and each arc in \( U \) meets \( J_i \), for \( i = 1, 2, \ldots, N \), in at most two partition elements. Thus, if \( y \in U \), then \( \text{diam}_{d_i}(J_i \cap [x, y]) \leq \varepsilon/4N \). We then have

\[
\rho^*(x, y) = \rho(x, y) + \sum_{i=1}^{N} \text{diam}_{d_i}(J_i \cap [x, y]) + \sum_{i=N+1}^{\infty} \text{diam}_{d_i}(J_i \cap [x, y])
\]

\[
< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon
\]

Therefore, \( U \subset B_{\rho^*}(x, \varepsilon) \). \( \square \)

In the following lemma, note that we are not claiming that the metric \( d_\rho \) is equivalent to \( \rho^* \) on \( X \).

2.3. Lemma. Let \( \rho^* \) be a bounded metric on \( X \) strictly monotone on arcs, and let \( p \in X \). For all \( x, y \in X \), define

\[
d_\rho(x, y) = \rho^*(p, x) + \rho^*(p, y) - 2\rho^*(p, x \wedge y).
\]

Then \( d_\rho \) is a convex metric on \( X \) bounded by twice the bound on \( \rho^* \).

Proof. Throughout the proof, \( \wedge \) will denote \( \wedge_p \). Since \( x \wedge y = y \wedge x \), \( d_\rho \) is symmetric. Since \( x \wedge y \in [p, x] \) and \( x \wedge y \in [p, y] \) and \( \rho^* \) is monotone on arcs, we have \( \rho^*(p, x) - \rho^*(p, x \wedge y) > 0 \) and \( \rho^*(p, y) - \rho^*(p, x \wedge y) > 0 \). Consequently, \( d_\rho(x, y) \geq 0 \). If \( x = y \) then \( x = x \wedge y = y \); so \( d_\rho(x, y) = 0 \). Conversely, if \( d_\rho(x, y) = 0 \), then \( \rho^*(p, x) - \rho^*(p, x \wedge y) = 0 \) and \( \rho^*(p, y) - \rho^*(p, x \wedge y) = 0 \) since their sum is 0 and both are nonnegative. Thus,

\[
\rho^*(p, x) = \rho^*(p, x \wedge y) \quad \text{and} \quad \rho^*(p, y) = \rho^*(p, x \wedge y).
\]

Since \( \rho^* \) is strictly monotone on arcs, this implies \( x = x \wedge y = y \). That \( d_\rho \) is bounded by twice the bound on \( \rho^* \) is clear.

To establish the triangle inequality, suppose \( x, y, z \in X \). Let \( v = x \wedge z \) and let \( b \) be the last point of the arc \( [p, y] \) that lies in \( [p, x] \cup [p, z] \). There are three cases: \( b \in [p, v] \), \( b \in (v, z] \), and \( b \in (v, x) \). (The reader may find it helpful to draw some figures.) We do the first case below and leave the similar proofs of the remaining cases to the reader.

If \( b \in [p, v] \), then \( x \wedge y = b = y \wedge z \), \( \rho^*(p, y) \geq \rho^*(p, b) \), and \( \rho^*(p, v) \geq \rho^*(p, b) \). Thus we have

\[
d_\rho(x, y) + d_\rho(y, z) = \rho^*(p, x) + 2 \rho^*(p, y) + \rho^*(p, z) - 4 \rho^*(p, b)
\]

\[
\geq \rho^*(p, x) + \rho^*(p, z) - 2 \rho^*(p, b)
\]

\[
\geq \rho^*(p, x) + \rho^*(p, z) - 2 \rho^*(p, v) = d_\rho(x, z).
\]

To verify that \( d_\rho \) is convex, let \( A \) be an arc in \( X \) and let \( x, y, z \in A \) such that \( y \in [x, z] \). Without loss of generality, assume that \( x \wedge z = v \in [y, z] \). Then \( x \wedge y = y \) and \( y \wedge z = v \). Thus we have

\[
d_\rho(x, y) = \rho^*(p, x) + \rho^*(p, y) - 2 \rho^*(p, y) = \rho^*(p, x) - \rho^*(p, y),
\]

\[
d_\rho(y, z) = \rho^*(p, y) + \rho^*(p, z) - 2 \rho^*(p, v),
\]

\[
d_\rho(x, y) + d_\rho(y, z) = \rho^*(p, x) + \rho^*(p, z) - 2 \rho^*(p, v) = d_\rho(x, z). \quad \square
\]
2.4. Example. The following example shows that it is not generally the case that if \( p^* \) is a metric strictly monotone on arcs and \( p \) is an arbitrary point in \( X \), then \( d_p \), defined as in Lemma 2.3, is equivalent to \( p^* \). However, it does follow by the argument in paragraph three of §2.5 that \( p^* \) is finer than \( d_p \).

Let \( p \) be the origin \((0, 0)\), \( x \) the point \((0, 2)\), and \([p, x]\) the line segment joining them in the plane. Let \( a_i = (0, 2 - \frac{1}{i}) \), \( x_i = (1, 2 - \frac{1}{i}) \), and let \([a_i, x_i]\) be the line segment joining them. Our space is

\[
X = [p, x] \cup \left( \bigcup_{i=1}^{\infty} [a_i, x_i] \right).
\]

The idea is to define a metric \( p^* \) on \( X \), strictly monotone on arcs, that has the property that from the point of view of \( p \) the arcs \([a_i, x_i]\) look short, but from the point of view of \( x \) they look long.

Let \( d \) denote the standard (Euclidean) metric on the plane, with \( \pi_1 \) and \( \pi_2 \) denoting the projections to the horizontal and vertical axes, respectively. Let \( d_1 \) denote the “arc-length” metric on \( X \); that is, the \( d_1 \)-distance between two points \( a, b \in X \) is the Euclidean arc-length of the unique arc \([a, b]\) between them. Let \( d_2 \) be defined as follows:

\[
d_2(b, a) = d_2(a, b) = \begin{cases} 
  d(a, b), & \text{if } a, b \in [p, x], \\
  d(a, a_i) + \frac{d(a_i, b)}{i} & \text{if } a \in [p, x], \; \pi_2(b) = \pi_2(a_i) \\
  \end{cases} 
\]

Note that \( d_2 \) is only defined on a subset of \( X \times X \).

The metric \( p^* \) on \( X \) is defined as follows:

\[
p^*(b, a) = p^*(a, b) = \begin{cases} 
  d_1(a, b), & \text{if } a, b \in [p, a_1] \text{ or } a, b \in X - [p, a_1], \\
  d_1(a, b) + (1-t)d_2(a, b), & \text{if } a \in [p, a_1], \; b \in X - [p, a_1], \; t = \pi_2(a). \\
  \end{cases} 
\]

We show below that \( p^* \) is a metric on \( X \), strictly monotone on arcs, and that the convex metric \( d_p \) on \( X \) defined with respect to \( p^* \) and \( p \) as in Lemma 2.3 has the property that \( d_p(x, x_i) \to 0 \). Since \( p^*(x, x_i) = d_1(x, x_i) > 1 \), for all \( i \), this shows \( d_p \) is not equivalent to \( p^* \).

It is easy to see that as a function from \( X \times X \) to \( \mathbb{R} \), \( p^* \) is nonnegative, symmetric, and takes value 0 if and only if \( a = b \). Moreover, on \([p, x]\) and on \( X - [p, a_1] \), \( p^* \) reduces to the arc-length metric \( d_1 \).

To see that \( p^* \) is strictly monotone on arcs, suppose \( a, b \in X \). If \( a, b \) are both in \([p, x]\) or \( X - [p, a_1] \), then \( p^* = d_1 \); hence, \( p^* \) is strictly monotone on \([a, b]\). So we may suppose \( a \in [p, a_1] \) and \( b \in (a_i, x_i) \), for some \( i \). Then (by abuse of notation) we see that \( 2 > \pi_2(b) = \pi_2(a_i) = a_i \geq 1 \), \( 0 \leq \pi_2(a) = a < 1 \), and \( 0 < \pi_1(b) = b \leq 1 \). One may then compute that

\[
p^*(a, b) = ad_1(a, b) + (1-a)d_2(a, b) \\
= a[d(a, a_i) + d(a_i, b)] + (1-a) \left[ d(a, a_i) + \frac{d(a_i, b)}{i} \right] \\
= a_i - a + b \left( a + \frac{1-a}{i} \right) \quad \text{written as a function of } b \\
= a_i + \frac{b}{i} + a \left( -1 + b - \frac{b}{i} \right) \quad \text{written as a function of } a.
\]
It is then easy to see that for fixed \( a \), as \( b \) increases (i.e., \([a, b]\) gets longer in the \( b \)-direction), \( \rho^*(a, b) \) increases, while for fixed \( b \), as \( a \) decreases (i.e., \([a, b]\) gets longer in the \( a \)-direction), \( \rho^*(a, b) \) increases. Therefore, \( \rho^* \) is strictly monotone on arcs.

Note that for \( a = 1, \ b = 0 \), or \( i = 1 \), \( \rho^*(a, b) \) reduces to the arc-length of \([a, b]\): \( d_1(a, b) = a_i - a + b \). The above arguments show that for all \( a, b \in X \), \( \rho^*(a, b) \leq d_1(a, b) \), with equality holding if either \( a, b \in [p, x] \cup [a_1, x_1] \) or \( a, b \in X - [p, a_1] \).

It remains to show that \( \rho^* \) satisfies the triangle inequality. So suppose \( a, b \in X \). Since \( \rho^* \) is strictly monotone on arcs, it suffices to check the triangle inequality for \( c \in [a, b] \). If \([a, b] \subset [p, x] \) or \([a, b] \subset X - [p, a_1] \), then \( \rho^* = d_1 \); so we may suppose \( a \in [p, a_1] \) and \( b \in (a_1, x_1] \), for some \( i \). Let \( c \in [a, b] \). As above, assume \( \pi_2(a_i) = a_i, \ \pi_1(b) = b, \) and \( \pi_2(a) = a \). There are three cases to consider.

If \( c \in [a_1, a_i] \), then we have
\[
\rho^*(a, b) \leq d_1(a, b) \leq d_1(a, c) + d_1(c, b) = \rho^*(a, c) + \rho^*(c, b).
\]
Suppose \( c \in [a, a_1] \); then \( 0 \leq a \leq \pi_2(c) = c \leq 1 \). Observe that
\[
\rho^*(a, b) = a_i - a + \frac{b}{i} + a \left( b - \frac{b}{i} \right),
\quad \rho^*(a, c) + \rho^*(c, b) = a_i - a + \frac{b}{i} + c \left( b - \frac{b}{i} \right).
\]
Since \( c \geq a \), the triangle inequality holds.

Finally, suppose \( c \in [a_i, b] \); then \( 0 \leq \pi_1(c) = c \leq b \leq 1 \). Observe that
\[
\rho^*(a, b) = a_i - a + b \left( a + \frac{1 - a}{i} \right),
\quad \rho^*(a, c) + \rho^*(c, b) = a_i - a + b + c \left( -1 + a + \frac{1 - a}{i} \right).
\]
The equation for \( \rho^*(a, c) + \rho^*(c, b) \) is linear in \( c \); when \( c = 0 \), it reduces to
\[
\rho^*(a, c) + \rho^*(c, b) = a_i - a + b = d_1(a, b) \geq \rho^*(a, b);
\]
when \( c = b \), it reduces to
\[
\rho^*(a, c) + \rho^*(c, b) = \rho^*(a, b).
\]
So for \( 0 \leq c \leq b \), the triangle inequality holds.

Now \( \rho^* \) is a metric on \( X \), strictly monotone on arcs. We define the convex metric \( d_p \) with respect to \( \rho^* \) and point \( p \in X \) as in Lemma 2.3. Then one may compute
\[
d_p(x_1, x_i) = 2 - d(p, a_i) + \frac{d(a_i, x_i)}{i} = \frac{2}{i}, \quad i \to 0.
\]

2.5. Proof of Theorem 1.2. Let \((X, r)\) be a uniquely arcwise connected, locally arcwise connected, separable metric space. Without loss of generality, suppose that \( r \) is bounded. Let \( \rho^* \) be the bounded metric of Lemma 2.2 equivalent to \( r \) and strictly monotone on arcs. In order to obtain a convex metric on \( X \) equivalent to \( r \), we must again use the separability of \( X \). Let \( \{p_i\}_{i=1}^\infty \) be a
countable dense set in $X$, and for each $i$, let $d_i = d_{p_i}$ be the convex metric of Lemma 2.3 defined in terms of $\rho^*$ and $p_i$. Note that there is a uniform bound on the $d_i$'s.

For all $x, y \in X$, define

$$d(x, y) = \sum_{i=1}^{\infty} \frac{d_i(x, y)}{2^i}.$$ 

It is easy to see that $d$ is a bounded convex metric on $X$. So it remains to show that $d$ is equivalent to $\rho^*$.

To see that $\rho^*$ is finer than $d$, observe that if $\rho^*(x_j, x) \to 0$, then, since $X$ is locally arcwise connected, $\rho^*(x_j \wedge_{p_i} x, x) \to 0$ for each $i$. Consequently, by the triangle inequality and the continuity of $\rho^*$, we have for each $i$,

$$\rho^*(p_i, x) - \rho^*(p_i, x_j \wedge_{p_i} x) \to 0,$$

$$\rho^*(p_i, x_j) - \rho^*(p_i, x_j \wedge_{p_i} x) \to 0.$$

Since

$$d_i(x_j, x) = \rho^*(p_i, x_j) - \rho^*(p_i, x_j \wedge_{p_i} x) + \rho^*(p_i, x) - \rho^*(p_i, x_j \wedge_{p_i} x),$$

it follows that $d_i(x_j, x) \to 0$ for each $i$. Since the sum converges, $d(x_j, x) \to 0$.

To prove that $d$ is finer than $\rho^*$, we show that for each $x \in X$ and each $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho^*(x, y) < \epsilon$. Let $\epsilon > 0$ and $x \in X$ be given. Choose $n$ so that $\rho^*(p_n, x) < \epsilon/3$. Choose $\delta = \epsilon/3(2^n)$ so that

$$d(x, y) < \delta \text{ implies } d_n(x, y) < \epsilon/3.$$ 

Now, by definition of $d_n$,

$$d_n(x, y) + 2\rho^*(p_n, x \wedge_{p_n} y) = \rho^*(p_n, x) + \rho^*(p_n, y).$$

Since $\rho^*$ is monotone on arcs, we have

$$\rho^*(p_n, x \wedge_{p_n} y) \leq \rho^*(p_n, x) < \epsilon/3.$$ 

Therefore,

$$\rho^*(x, y) \leq \rho^*(p_n, x) + \rho^*(p_n, y) = d_n(x, y) + 2\rho^*(p_n, x \wedge_{p_n} y) < \epsilon/3 + 2\epsilon/3 = \epsilon.$$ 

This concludes the proof of the main theorem.

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