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ON THE ABSTRACT CAUCHY PROBLEM IN FRÉCHET SPACES

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ABSTRACT. Let A be a linear operator in a Fréchet space X with the resolvent defined for all $\lambda > 0$. In this note we prove that both the first and the second order abstract Cauchy problems associated to A are well posed on certain maximal subspaces of X. Our results extend those of Kantorovitz [5] and Cioranescu [2].

1. INTRODUCTION

The abstract Cauchy problem (abbreviated, ACP) associated to a linear operator A has been considered by many authors. For the most part of these works, A is an operator defined in a Banach space X, but the situation whenever X is a locally convex space also has been studied [3, 4, 6, 7].

The interest of the ACP in Fréchet spaces has been enhanced by the works of Ujishima [10], Cioranescu [1], and recently Watanabe [11]. In [10, 1] the ACP in Fréchet spaces has been related with the ACP in the sense of distributions whilst in [11] the ACP of the second order has been resolved in certain linear topological space included in X.

Let X be a Banach space and A be a linear operator in X such that the resolvent operator $R(\lambda, A) = (\lambda - A)^{-1}$ is a bounded operator on X for all $\lambda > 0$. Recently S. Kantorovitz [5], for the first order case, and I. Cioranescu [2] for the second one, have showed that the ACP is well posed in a maximal subspace of X.

The aim of this note is to show that these results are also true when X is a Fréchet space. In order to prove this assertion we will use the properties of strongly continuous semigroup of operators and strongly continuous cosine functions of operators. In particular, the first order ACP is well posed if and only if A is the infinitesimal generator of a strongly continuous semigroup of operators whilst the second order ACP is well posed if and only if A is the infinitesimal generator of a cosine function of operator [13]. Our proofs are an adaptation of those carried out by Kantorovitz in [5].

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Let X be a Fréchet space. It is well known [9, Theorem 1.24] that the topology of X is determined by a metric d that satisfies the following properties:

(a) The metric d is invariant under translations.
(b) The open balls centered at 0 are balanced.
(c) The open balls are convex.

In this case we will say that d satisfies the condition (F).

Throughout this paper we will denote by X a Fréchet space whose topology is determined by a metric d that satisfies the condition (F). For an arbitrary linear operator A with domain $D(A)$ in X we denote by $\rho(A)$ the resolvent set of A. We will say that a bounded linear operator U defined on X commutes with A if $U(D(A)) \subseteq D(A)$ and $UAx = AUx$, for every $x \in D(A)$. For the operators A such that $\rho(A) \neq \Phi$ we know that U commutes with A if and only if U commutes with the resolvent operator $R(\lambda, A)$, for some $\lambda \in \rho(A)$. Moreover, if Z is a subspace of X then A_z, called the part of A in Z, will denote the restriction of A to the domain $D(A_z) = \{x \in D(A) \cap Z : Ax \in Z\}$. Furthermore, if Z is a Fréchet space contained in X we use the notation $Z \hookrightarrow X$ to represent that the inclusion of Z into X is continuous. Finally, we denote by $\mathscr{L}(X)$ the space of bounded linear operators on X.

2. The first order abstract Cauchy problem

Let A be a linear operator in X with domain $D(A)$. We assume that every $\lambda > 0$ is included in the resolvent set $\rho(A)$ of A, and we introduce the set $\mathscr{P} = \{\prod_{j=1}^{k} \lambda_j R(\lambda_j, A) : k \in \mathbb{N}, \lambda_j > 0\} \cup \{I\}$.

Let Φ be the function defined by

$$\Phi(x, y) = \sup\{d(Px, Py) : P \in \mathscr{P}\},$$

for every $x, y \in X$. Let Y be the set of elements $x \in X$ such that $\Phi(x, 0) < +\infty$. We begin by summarizing some elementary properties of Φ and Y.

Proposition 2.1. *The following properties hold.*

(i) *The set Y is a linear subspace of X, and the function Φ on Y is a metric that satisfies the condition* (F). *Furthermore, the metric Φ majorizes the metric d and for every $P \in \mathscr{P}$ and $x, y \in Y$,*

$$\Phi(Px, Py) \leq \Phi(x, y). \tag{1}$$

(ii) *The set Y with the topology induced by Φ is a Fréchet space.*

(iii) *The space Y is invariant for every continuous linear operator U that commutes with A.*

Proof. The assertion (i) is an easy consequence of the properties of the metric d and the definition of Φ. In particular, the relation (1) follows from the fact that $\mathscr{P}$ is closed in $\mathscr{L}(X)$ for the composition of operators.

On the other hand, for the properties of Φ it follows that Y is a locally convex space. To prove the completeness, we consider a Cauchy sequence $(y_n)_n$ in Y. It is clear that $(y_n)_n$ is also a Cauchy sequence in X that satisfies the condition: given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\Phi(y_n, y_m) = \sup_{P \in \mathscr{P}} d(Py_n, Py_m) \leq \varepsilon, \tag{2}$$

for every $m, n \geq n_0$. Let x be the limit of $(y_n)_n$ in X. Then, for each $P \in \mathscr{P}$ and $n \geq n_0$, from (2) we obtain

$$d(Py_n, Px) = \lim_{m\to\infty} d(Py_n, Py_m) \leq \varepsilon,$$

which proves that $x \in Y$ and the sequence $(y_n)_n$ converges to x in the space Y. This completes the proof of (ii).

Finally, if U is a bounded linear operator that commutes with A then U commutes with every operator $P \in \mathscr{P}$. Consequently, for every $x \in Y$ the set $\{PUx: P \in \mathscr{P}\} = \{UPx: P \in \mathscr{P}\}$ is bounded for the metric d. This implies that $\Phi(Ux, 0) < +\infty$ and $Ux \in Y$.

We will also need the following result.

Lemma 2.1. *Let W be a Fréchet space and B be a linear operator in W such that every $\lambda > 0$ is included in $\rho(B)$ and the set of operators $\{\lambda^n R(\lambda, B)^n: \lambda > 0,\ n \in \mathbb{N}\}$ is equicontinuous. Then there exists a metric d_1 on W that is compatible with the topology of W, satisfies the condition* (F), *and*

$$d_1(\lambda R(\lambda, B)w, 0) \leq d_1(w, 0)$$

for every $\lambda > 0$ and $w \in W$.

Proof. We may assume that the topology of W is determined by a set $\{p_n: n \in \mathbb{N}\}$ of seminorms, which may be chosen so that $p_n(x) \leq p_{n+1}(x)$, for all $x \in W$ and $n \in \mathbb{N}$.

Since $\{\lambda^m R(\lambda, B)^m: \lambda > 0 \text{ and } m \in \mathbb{N}\}$ is an equicontinuous set of bounded operators, for each $n \in \mathbb{N}$ there exists a positive constant M_n such that

$$p_n(\lambda^m R(\lambda, B)^m x) \leq M_n p_n(x)$$

for all $\lambda > 0$, $m \in \mathbb{N}$ and every $x \in W$.

On the other hand, the argument used in the proof of Lemma 5.1, [8, Chapter I] also applies to seminorms. Therefore, we may affirm that, for each $n \in \mathbb{N}$, there exists a seminorm q_n on W that is equivalent to the seminorm p_n and satisfies the properties

$$p_n(x) \leq q_n(x) \leq M_n p_n(x), \quad \forall x \in W \tag{3}$$

and

$$q_n(\lambda R(\lambda, B)x) \leq q_n(x), \quad \forall x \in W,\ \forall \lambda > 0. \tag{4}$$

Let V_n be the open ball $\{x \in W: q_n(x) < 1/2^n\}$; then $\{V_n: n \in \mathbb{N}\}$ is a base at 0 formed by balanced sets such that $V_{n+1} + V_{n+1} \subseteq V_n$. In these conditions we may apply the proceeding used in the proof of Theorem 1.24 in [9]. The metric d_1 obtained in this form satisfies all the statements of the lemma. In fact, the condition (F) follows directly from the result already mentioned. Hence it remains only to prove (4). Let $A(r)$ and f be as in the construction carried out in [9, Theorem 1.24]. For each $x \in W$, if $f(x) < r$, with

$$r = \sum_{n=1}^{n_0} c_n(r) 2^{-n}$$

then $x \in A(r)$. This means that there exists $v_n \in V_n$, for every $n = 1, 2, \ldots, n_0$ such that $x = \sum_{n=1}^{n_0} c_n(r) v_n$. Consequently, for each $\lambda > 0$,

$$\lambda R(\lambda, B)x = \sum_{n=1}^{n_0} c_n(r) \lambda R(\lambda, B) v_n.$$

But, since

$$q_n(\lambda R(\lambda, B)v_n) \leq q_n(v_n) < 1/2^n,$$

it follows that $\lambda R(\lambda, B)v_n \in V_n$ and $\lambda R(\lambda, B)x \in A(r)$. Now, from the definition of d_1 we obtain that

$$d_1(\lambda R(\lambda, B)x, 0) = f(\lambda R(\lambda, B)x) \leq r.$$

Hence

$$d_1(\lambda R(\lambda, B)x, 0) \leq f(x) = d_1(x, 0),$$

which completes the proof. □

We now state the main result of this section, which is an extension to Fréchet spaces of Theorem 2.4 in [5]. In this statement Z will denote the space $\overline{D(A_Y)}$, where the closure is taken in (Y, Φ).

Theorem 2.1. *The operator A_z is the infinitesimal generator of an equicontinuous semigroup of class C_0 on Z. Moreover, Z is maximal in the following sense: If W is a Fréchet space that $W \hookrightarrow X$ and the operator A_W generates an equicontinuous semigroup of class C_0 on W then $W \hookrightarrow Z$ and $D(A_W) \subseteq D(A_Z)$.*

Proof. We start by showing that $\lambda \in \rho(A_z)$, for every $\lambda > 0$. If $x \in D(A_Y)$ then $x \in D(A) \cap Y$ and $Ax \in Y$. Since $R(\lambda, A)$ is a bounded linear operator, it follows that $R(\lambda, A)x \in D(A) \cap Y$; and since $AR(\lambda, A)x = \lambda R(\lambda, A)x - x$ belongs to Y, we obtain that $R(\lambda, A)(D(A_Y)) \subseteq D(A_Y)$. Now, using the continuity of the operator $R(\lambda, A)$ with respect to the metric Φ, we infer that

$$R(\lambda, A)(Z) = R(\lambda, A)(\overline{D(A_Y)}) \subseteq \overline{D(A_Y)} = Z.$$

Consequently $\lambda \in \rho(A_z)$ and $R(\lambda, A_z) = R(\lambda, A)|_Z$.

Next we will prove that the domain of A_z is dense in Z. Let $y \in D(A_Y)$. Then $\lambda R(\lambda, A)y \to y$, as $\lambda \to +\infty$, for the metric Φ. In fact, from the properties of Φ it follows that

$$\begin{aligned}\Phi(\lambda R(\lambda, A)y - y, 0) &= \Phi(R(\lambda, A)Ay, 0) \\ &= \Phi(\lambda R(\lambda, A), Ay/\lambda, 0) \leq \Phi(Ay/\lambda, 0),\end{aligned}$$

which shows the assertion. This implies that $D(A_Y^2)$ is dense in $D(A_Y)$. It is easy to see that $D(A_Y^2)$ is included in $D(A_Z)$ so that $D(A_Z)$ is dense in Z.

On the other hand, for every $n \in \mathbb{N}$, $\lambda > 0$, and every $x, y \in Z$, it follows from (1) that

$$\Phi(\lambda^n R(\lambda, A_Z)^n x, \lambda^n R(\lambda, A_Z)^n y) = \Phi(Px, Py) \leq \Phi(x, y),$$

where $P = \lambda^n R(\lambda, A)^n$. Therefore, the set of operators $\{\lambda^n R(\lambda, A_z)^n : \lambda > 0, n \in \mathbb{N}\}$ is equicontinuous and from the Miyadera theorem [7, Theorem 5.2] we may assert that A_Z is the infinitesimal generator of an equicontinuous semigroup of class C_0 on Z.

We will now prove the maximality property of Z. In the remainder of the proof, we suppose that d_1 is a metric on W that satisfies all of the conditions established in Lemma 2.1. From the Miyadera theorem [7] we know that the interval $(0, +\infty) \subseteq \rho(A_W)$, and it is clear that $R(\lambda, A_W)w = R(\lambda, A)w$, for every $\lambda > 0$ and $w \in W$. Hence, if $P = \prod_{j=1}^{k} \lambda_j R(\lambda_j, A)$ is an element of $\mathscr{P}$ then $P(W) \subseteq W$ and $Pw = \prod_{j=1}^{k} \lambda_j R(\lambda_j, A_W)w$, for each $w \in W$. Since

$$d_1(Pw, 0) = d_1\left(\prod_{j=1}^{k} \lambda_j R(\lambda_j, A)w, 0\right) \leq d_1(w, 0),$$

it follows that the set $\{Pw : P \in \mathscr{P}\}$ is bounded in W. Knowing that W is continuously included in X we conclude that $\{Pw : P \in \mathscr{P}\}$ is a bounded set in X, for all $w \in Y$. This shows that $W \subseteq Y$. The same argument allow us to obtain that if S is a bounded subset of W, then the set $\{Pw : P \in \mathscr{P}, w \in S\}$ is also bounded in X. Thus, S is bounded in Y for the metric Φ, which implies that W is continuously included in Y. Since $D(A_W) \subseteq D(A_Y)$ and $D(A_W)$ is dense in W, we obtain that

$$W = \overline{D(A_W)}^{W} \subseteq \overline{D(A_W)}^{Y} \subseteq \overline{D(A_Y)}^{Y} = Z.$$

This finally yields $W \hookrightarrow Z$ and $D(A_W) \subseteq D(A_Z)$.

Remark. Concerning the nontriviality of space Z, we may prove that Z contains the eigenvectors corresponding to eigenvalues α of A with $\operatorname{Re}(\alpha) \leq 0$. In fact, if $Ax = \alpha x$ then $R(\lambda; A)x = (\lambda - \alpha)^{-1}x$, for every $\lambda > 0$. Thus, for every $P = \prod_{j=1}^{k} \lambda_j R(\lambda_j, A)$ we obtain

$$d(Px, 0) = d\left(\prod_{j=1}^{k} \frac{\lambda_j}{\lambda_j - \alpha} x, 0\right) \leq d(x, 0),$$

which implies that $x \in Y$. Since $x \in D(A_Y)$ then $x \in Z$.

Example. As an example that shows how Theorem 2.1 can be applied, we consider the space $X = C(-\infty, 0; \mathbb{C})$ provided with the compact-open topology and let L be a continuous linear form on X such that the function $f(\lambda) = L(e^{\lambda\theta})$ satisfies the following conditions:

(a) $f(0) = 0$;
(b) $f(\lambda) \neq \lambda$, for every $\lambda > 0$;
(c) $\lambda/(\lambda - f(\lambda)) \to +\infty$, as $\lambda \to 0^+$, $\lambda \in \mathbb{R}$.

Let A be the operator defined by

$$Ax(\theta) = dx/d\theta$$

on $D(A) = \{x \in C^1(-\infty, 0; \mathbb{C}) : x'(0) = L(x)\}$.

It is easy to see that $\lambda = 0 \in \sigma_p(A)$ so that the space $Z \neq \{0\}$. Furthermore the interval $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$ the resolvent operator $R(\lambda, A)$ is given by

$$R(\lambda, A)g(t) = \frac{e^{\lambda t}}{\lambda - f(\lambda)}\left\{g(0) - L\left[\int_0^{\theta} e^{\lambda(\theta-\tau)} g(\tau)d\tau\right]\right\} - \int_0^{t} e^{\lambda(t-\tau)} g(\tau)d\tau$$

for every $g \in X$ and $t \leq 0$. Therefore $\{\lambda R(\lambda, A): \lambda > 0\}$ is not an equicontinuous set of bounded operators, which shows that $Z \neq X$.

A specific example is given by the linear form $L(\varphi) = \varphi(a) - \varphi(a-1)$, where $a < 0$.

3. The second order abstract Cauchy problem

Let A be a linear operator in X with domain $D(A)$. We assume that the resolvent $R(\lambda, A)$ is a bounded operator on X for every $\lambda > 0$. We will use the notation $F(\lambda) = \lambda R(\lambda^2, A)$ and introduce the set of operators $\mathscr{P} = \{(\lambda^{n+1}/n!)F(\lambda)^{(n)}: \lambda > 0,\ n \in \mathbb{N}\} \cup \{I\}$.

Proceeding as in §2, let Φ be the function defined by

$$\Phi(x, y) = \sup\{d(Px, Py): P \in \mathscr{P}\}$$

for every $x, y \in X$, and let Y be the set $\{x \in X: \Phi(x, 0) < +\infty\}$. It is easy to see that this definition has the same properties established in Proposition 2.1. Therefore, we omit it the proof.

Proposition 3.1. *The properties established in Proposition* 2.1 *hold.*

For the next theorem we need the following result.

Lemma 3.1. *Let W be a Fréchet space and B be a linear operator in W, which generates an equicontinuous cosine function of class C_0 on W. Let $(p_n)_n$ be a family of seminorms that generates the topology of W. For every $\lambda > 0$, we set $F(\lambda) = \lambda R(\lambda^2, B)$. Then there exists a family of seminorms $(q_n)_n$ and a sequence of constants $(M_n)_n$ such that*

$$p_n(x) \leq q_n(x) \leq M_n p_n(x) \tag{5}$$

and

$$q_n\left(\frac{\lambda^{n+1}}{n!}F^{(n)}(\lambda)x\right) \leq q_n(x), \tag{6}$$

for every $n \in \mathbb{N}$, $\lambda > 0$, and $x \in W$.

Proof. Let C be the cosine function of operators generated by B. Since the set $\{C(t): t \in \mathbb{R}\}$ is equicontinuous on W, for each $n \in \mathbb{N}$ there exists $M_n \geq 0$ such that

$$p_n(C(t)x) \leq M_n p_n(x) \tag{7}$$

for every $x \in W$ and $t \in \mathbb{R}$. Thus, we may define

$$q_n(x) = \sup\{p_n(C(t)x): t \in \mathbb{R}\} \tag{8}$$

for each $x \in W$ and $n \in \mathbb{N}$. Clearly q_n is a seminorm in W and (5) follows from (7) and the condition $C(0) = I$.

On the other hand, $q_n(C(s)x) \leq q_n(x)$, for all $s \in \mathbb{R}$ and $x \in W$. In fact, using the properties of the cosine functions we may write

$$\begin{aligned} q_n(C(s)x) &= \sup\{p_n(C(t)C(s)x): t \in \mathbb{R}\} \\ &= \sup\{p_n(\tfrac{1}{2}C(t+s)x + \tfrac{1}{2}C(t-s)x): t \in \mathbb{R}\} \\ &\leq \tfrac{1}{2}\sup\{p_n(C(u)x): u \in \mathbb{R}\} + \tfrac{1}{2}\sup p_n(C(v)x): v \in \mathbb{R} = q_n(x). \end{aligned}$$

It is clear that the family of seminorms q_n, defined according to the previous procedure, generates the topology of W. Now proceeding as in the proof of Theorem 2.1 in [3] we obtain (6).

It is easy to see that the same arguments used in the proof of Lemma 2.1 allow us to conclude that there exists a metric d_1 compatible with the topology of W that verifies the condition (F) and

$$d_1\left(\frac{\lambda^{n+1}}{n!}F^{(n)}(\lambda)x\,,\,0\right) \leq d_1(x\,,\,0) \tag{9}$$

for every $n \in \mathbb{N}$, $\lambda > 0$, and $x \in W$.

In the next result V will denote the space $\overline{D(A_Y)}$, where the closure is taken in $(Y\,,\,\Phi)$.

Theorem 3.1. *The operator A_V is the infinitesimal generator of an equicontinuous cosine function on V. Furthermore, V is maximal in the following sense: if W is a Fréchet space such that $W \hookrightarrow X$ and the operator A_W is the infinitesimal generator of an equicontinuous cosine function on W then $W \hookrightarrow V$ and $D(A_W) \subseteq D(A_V)$.*

Proof. The demonstration in this case is very similar of that carried out in the Theorem 2.1. By this reason we shall only sketch the proof. Initially, using a theorem due to Fattorini [3] instead of the theorem of Miyadera we may prove that A_V is the infinitesimal generator of an equicontinuous cosine functions of operators on V.

Next, in order to prove the maximality property of V we use Lemma 3.1 with the operator $B = A_W$. Thus, we may assume defined on W a metric d_1 that satisfies the condition (F) and the inequality (9). Now, the same argument used in the proof of Theorem 2.1 allows us to conclude that $W \hookrightarrow V$ and $D(A_W) \subseteq D(A_V)$.

Concerning this theorem, it is worth mentioning that V is included in Z. In fact, it is well known [3] that every infinitesimal generator of a strongly continuous cosine function is also a generator of a strongly continuous semigroup of operators.

On the other hand, concerning the nontriviality of V, we may prove that it contains the eigenvectors corresponding to nonpositive eigenvalues of A.

Remark. We have already mentioned that the well posedness of the ACP of first and second order, in a Fréchet space X, is directly related to the theory of one parameter semigroups and cosine functions of operators, respectively. However, this relation is also true when X is a sequentially complete locally convex space (see Konishi [6] and Yosida [12]). By this reason, we expect that our results can be extended to include locally equicontinuous semigroups and cosine functions of operators on sequentially complete locally convex spaces.

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