RADON-NIKODYM PROPERTY IN SYMMETRIC SPACES OF MEASURABLE OPERATORS

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ABSTRACT. Let $E$ be a rearrangement invariant function space on $(0, \infty)$ with the RNP. Let $(M, \tau)$ be a von Neumann algebra with a faithful normal semifinite trace $\tau$. It is proved that the associated symmetric space $L_E(M, \tau)$ of measurable operators has the RNP.

1. Introduction

The aim of this note is to solve a problem arising in a previous work [10] concerning the Radon–Nikodym property (RNP) in symmetric spaces of measurable operators. For stating our result, we introduce the necessary definitions and notations (for unexplained notions, see [5, 6, 9]). We also refer the reader to [2] for the definition and the elementary properties of the RNP.

Let $(M, \tau)$ be a semifinite von Neumann algebra acting on a Hilbert space $H$, with a faithful normal semifinite trace $\tau$. Let $\overline{M}$ be the space of all measurable operators with respect to $(M, \tau)$ in the sense of [7], equipped with the measure topology defined there. For $a \in \overline{M}$ and $t > 0$ the $t$th $s$-number (singular number) of $a$ is defined by (cf. [3])

$$\mu_t(a) = \inf \{ \|ae\| : e \text{ is a projection in } M \text{ with } \tau(1-e) \leq t \}.$$ 

The function $t \mapsto \mu_t(a)$ on $(0, \infty)$ is denoted by $\mu(a)$. This is a positive nonincreasing function on $(0, \infty)$. Note that $\mu(a) = \mu(a^*) = \mu(|a|)$, $|a|$ being the absolute value of $a$. Recall the following useful formula for $\mu_t(a)$ in terms of the spectral measures (cf. [3])

$$\mu_t(a) = \inf \{ s \geq 0 \mid \lambda_s(a) \leq t \},$$

where $\lambda_s(a) = \tau(e_{(s, \infty)}(|a|))$, $e_{(s, \infty)}(|a|)$ being the spectral projection of $|a|$ corresponding to the interval $(s, \infty)$ for $s \geq 0$. It follows immediately that

$$\mu_t(a) = 0 \text{ for } t \geq \tau(\text{supp}(a)),$$

where $\text{supp}(a)$, the support of $a$, is the smallest projection $e$ in $M$ such that $ae = a$. In particular, if $\tau$ is finite, i.e., $\tau(1) < \infty$, then for any $a \in \overline{M}$

$$\mu_t(a) = 0 \text{ for } t \geq \tau(1).$$

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Now let $E$ be a rearrangement invariant (r.i.) function space on $(0, \infty)$ in the sense of [6]. Then we define the symmetric space of measurable operators associated with $E$ and $(M, \tau)$ as follows:

$$L_E(M, \tau) = \{a \in \mathcal{M} | \mu(a) \in E\}$$

and

$$\|a\|_{L_E(M, \tau)} = \mu(a) = \|\mu(a)\|_E, \quad a \in L_E(M, \tau).$$

It is an elementary fact that $(L_E(M, \tau), \|\cdot\|)$ is a Banach space (cf. [10, Lemma 4.1]). Moreover, we have $\|a\|_E = \|a^*\| = \|\mu(a)\|_E$ for any $a \in L_E(M, \tau)$. If $E = L_p(0, \infty) (0 < p \leq \infty)$, then $L_E(M, \tau)$, denoted usually by $L_p(M, \tau)$ in this case, is simply the usual noncommutative $L_p$-space associated with a semifinite von Neumann algebra. Note that $L_\infty(M, \tau)$ is just $M$ equipped with the operator norm.

If $\tau$ is finite, by (2) we have $\mu_t(a) = 0$ for $t \in [\tau(1), \infty)$ ($a \in \mathcal{M}$); so that in this case $E$ can be taken as an r.i. function space on $(0, \tau(1))$.

It would be natural to expect properties of a symmetric space of measurable operators to reflect the properties of the corresponding r.i. function space. Thus we proved in [11] that uniform convexity and uniform PL-convexity pass from $E$ to $L_E(M, \tau)$, and in [10] that the same phenomenon occurs for the analytic Radon–Nikodym property and the uniform $H$-convexity in the sense of [12]. A related problem arising from [10] is to know whether the RNP also passes from $E$ to $L_E(M, \tau)$. We answer this question by the following result.

**Theorem.** Let $E$ be an r.i. function space on $(0, \infty)$ having the RNP. Then for any semifinite von Neumann algebra $(M, \tau)$, $L_E(M, \tau)$ possesses the RNP.

It is worthwhile to note that a similar result for unitary ideals is easy (of course, our theorem contains this special case). In that case, $E$ is a symmetric Banach sequence space on $\mathbb{N}$ and $(M, \tau)$ is just $(B(H), \text{tr})$, where $B(H)$ is the space of all bounded operators on a separable Hilbert space $H$ and $\text{tr}$ is the usual trace on $B(H)$. Traditionally, $L_E(B(H), \text{tr})$ is denoted by $C_E$, the unitary ideal associated with $E$. If $E$ has the RNP, $E$ fails to contain $C_0$. Then it follows easily that the canonical basis is a boundedly complete basis for $E$ with a basis constant 1. Therefore $E$ is order isometric to the dual of $E'$, $E'$ being the closed subspace of the dual of $E$ generated by all finite sequences (cf. [5]). Now by a well-known result from the theory of unitary ideals (cf. [4, Theorem III.12.2]), we have

$$(C_E')^* = C_{(E')} = C_E.$$

Note also that $C_E$ is separable because $E$ is separable. Hence $C_E$ is a separable dual, so it has the RNP. This simple reasoning also shows that for symmetric Banach sequence spaces and unitary ideals, the RNP is equivalent to the absence of $C_0$ in these spaces. Therefore in these spaces the RNP and the analytic RNP coincide.

For further results about symmetric spaces of measurable operators the reader is referred to [10].

**2. Proof**

Now we proceed to prove the theorem. Our proof relies heavily on the idea in the proof of the corresponding result for the analytic RNP in [10]. In the
following, $E$ always denotes an r.i. function space on $(0, \infty)$ and $(M, \tau)$, a semifinite von Neumann algebra on $H$, with a faithful normal semifinite trace $\tau$.

As in [10], we first consider the finite trace case.

**Lemma 1.** Suppose $\tau(1) = 1$. Let $F$ be an order continuous r.i. function space on $(0, 1)$. Then

$$(L_F(M, \tau))^* = L_{F^*}(M, \tau).$$

**Proof.** Note first that the order continuity of $F$ implies that $F^*$ consists only of measurable functions on $(0, 1)$ (cf. [6]). Hence $F^*$ is also an r.i. function space on $(0, 1)$, so $L_{F^*}(M, \tau)$ is well defined.

The inclusion

$$L_{F^*}(M, \tau) \subset (L_F(M, \tau))^* \quad \text{(of norm $\leq 1$)}$$

is easily seen. Indeed, by [3, Theorem 4.2], for any $b \in L_{F^*}(M, \tau)$ and any $a \in L_F(M, \tau)$, we have

$$\int_0^1 \mu_t(ab) \, dt \leq \int_0^1 \mu_t(a)\mu_t(b) \, dt.$$  

Thus $\tau(ab)$ is well defined and

$$|\tau(ab)| \leq \int_0^1 \mu_t(a)\mu_t(b) \, dt.$$  

It follows that the linear functional $l: a \mapsto \tau(ab)$ defined on $L_F(M, \tau)$ is continuous and of norm $\leq \|\mu(b)\|_{F^*} = \|b\|_{F^*}$.

For the converse inclusion we take $l \in (L_F(M, \tau))^*$. Then we must show that $l$ is defined by an element $b \in L_{F^*}(M, \tau)$ and that the norm of $l$ in $(L_F(M, \tau))^*$ is less or equal to $\|b\|_{L_{F^*}(M, \tau)}$. For this we first show that $l$ is in the predual of $M$, that is, $l$ is an element of $L_1(M, \tau)$ (it is well known that $(L_1(M, \tau))^* = L_\infty(M, \tau) = M$).

Because $F$ is an r.i. function space on $(0, 1)$, we have (cf. [6])

$$L_\infty(0, 1) \subset F \subset L_1(0, 1) \quad \text{(inclusions of norm $\leq 1$)}.$$  

It follows that

$$M \subset L_F(M, \tau) \subset L_1(M, \tau) \quad \text{(inclusions of norm $\leq 1$)}.$$  

Therefore the continuous linear functional $l$ on $L_F(M, \tau)$ also defined a continuous linear functional on $M$, that is, $l \in M^*$. In order to show that $l$ is in the predual of $M$, by a well-known result from the theory of operator algebras (cf. [9, Corollary III.3.11]), it suffices to show the following: For any orthogonal family $\{e_i\}_{i \in I}$ of projections in $M$

$$l \left( \sum_{i \in I} e_i \right) = \sum_{i \in I} l(e_i).$$

Equation (3) immediately follows from the following lemma.
Lemma 2. Let \( \{e_i\}_{i \in I} \) be an orthogonal family of projections of \( M \). Then \( \sum_{i \in I} e_i \) converges in \( L_F(M, \tau) \).

Proof. For each finite subset \( J \) of \( I \), let \( h_J = \sum_{i \in J} e_i \), and let \( k_J = \sum_{i \notin J} e_i \) (convergence in the strong operator topology). Then the decreasing net \( \{k_J : J \text{ finite subset of } I\} \) converges to 0 in the strong operator topology. Since \( \tau \) is normal and finite, the net \( \{\tau(k_J)\} \) converges to 0. Since \( F \) is order continuous, \( \{1_{[0,\tau(k_J)]} \} \rightarrow 0 \), \( 1_w \) being the indicator function of a subset \( w \). As \( \mu(k_J) = 1_{[0,\tau(k_J)]}, \{k_J\} \rightarrow 0 \) in \( L_F(M, \tau) \). This proves Lemma 2.

End of the proof of Lemma 1. Now \( l \) is in the predual of \( M \); namely, there exists a measurable operator \( b \in L_1(M, \tau) \) such that

\[
l(a) = \tau(ab), \quad \forall a \in M.
\]

Consequently

\[
|\tau(ab)| \leq ||l|| ||a||_F, \quad \forall a \in M.
\]

Now by [10, Lemma 4.5], \( M \) is dense in \( L_F(M, \tau) \); therefore, for any \( a \in L_F(M, \tau) \), \( \tau(ab) \) is well defined and (4) and (5) hold for \( a \in L_F(M, \tau) \). We next show that \( b \in L_{F^*}(M, \tau) \) and \( ||b||_{F^*} \leq ||l|| \).

Let \( b = u|b| \) be the polar decomposition of \( b \) and \( |b| = \int_0^\infty t \, d\mu_t \) be the spectral decomposition of \( |b| \). Let \( \hat{e}_t = e_{\mu_t(b) - 0} \) \((t > 0, \, e_{0-0} = 1)\). Then \( |b| \) admits the following Schmidt decomposition (cf. [8])

\[
|b| = \int_0^1 \mu_t(b) \, d\hat{e}_t.
\]

Let \( x \) be a nonincreasing positive function in \( F \) such that \( x \) is constant in the intervals where \( \mu(b) \) is constant. Define

\[
a = \left( \int_0^1 x(t) \, d\hat{e}_t \right) \cdot u^*.
\]

Then it is easy to check that \( a \in L_F(M, \tau) \) and \( ||a||_F \leq ||x||_F \). We also have

\[
\int_0^1 x(t) \mu_t(b) \, dt = \tau \left( \int_0^1 x(t) \, d\hat{e}_t \cdot \int_0^1 \mu_t(b) \, d\hat{e}_t \right) = \tau(ab).
\]

Then by (5)

\[
\int_0^1 x(t) \mu_t(b) \, dt \leq ||l|| \cdot ||a||_F \leq ||l|| \cdot ||x||_F.
\]

Taking the supremum in the above inequalities over all \( x \) satisfying the previous property and \( ||x||_F \leq 1 \), we deduce that \( \mu(b) \in F^* \) and \( ||\mu(b)||_{F^*} \leq ||l|| \). Consequently, \( b \in L_{F^*}(M, \tau) \) and \( ||b||_{L_{F^*}(M, \tau)} \leq ||l|| \). This concludes the proof of Lemma 1.

Lemma 3. Assume that \( \tau \) is finite and \( E \) has the RNP. Then \( L_E(M, \tau) \) has the RNP.

Proof. After a trivial normalization, we can assume \( \tau(1) = 1 \). Then we can regard \( E \) as an r.i. function space on \( (0, 1) \). The RNP of \( E \) implies that \( E \) is not isomorphic to \( L_1(0, 1) \) and that \( E \) is maximal in the sense of [6].
Consequently, \( E = F^* \), where \( F \) is an order continuous r.i. function space on \((0, 1)\) (cf. [6]). Then by Lemma 1

\[
L_E(M, \tau) = (L_E(M, \tau))^*.
\]

If \( L_E(M, \tau) \) was separable, then \( L_E(M, \tau) \), as a separable dual, would have the RNP. But unfortunately, \( L_E(M, \tau) \) is in general nonseparable. Therefore, we must do something else in order to prove the RNP of \( L_E(M, \tau) \).

Now let \( X \) be a separable closed subspace of \( L_E(M, \tau) \). We show that \( X \) is isometric to a closed subspace of a separable dual, from which [2, Corollary III.3.5] and Lemma 3 follows.

Let \( \{a_n\}_{n \geq 0} \) be a dense sequence in \( X \); by [10, Lemma 4.3], \( M \) is dense in \( L_E(M, \tau) \). Then by approximating \( a_n \) by elements in \( M \), we may assume \( a_n \in M \) for any \( n \geq 0 \). Now let \( \bar{M} \) be the von Neumann subalgebra of \( M \) generated by \( 1 \) and all the \( a_n \)'s. Let \( \bar{\tau} \) be the restriction of \( \tau \) to \( \bar{M} \). Clearly, \( \bar{\tau} \) is a faithful normal finite trace on \( \bar{M} \). It is also clear that \( L_E(M, \bar{\tau}) \) is naturally identified with a closed subspace of \( L_E(M, \tau) \) and \( X \) a closed subspace of \( L_E(M, \bar{\tau}) \). Now by Lemma 1, \( L_E(M, \bar{\tau}) \) is a dual. Furthermore, by [10, Lemma 5.6], \( L_E(M, \bar{\tau}) \) is separable. Thus, \( X \) is a closed subspace of the separable dual \( L_E(M, \bar{\tau}) \), proving Lemma 3.

Lemma 3 proves the theorem in the finite trace case. We reduce the general case to the finite trace case by using the semifiniteness of \( \tau \). The following argument is similar to the corresponding part in the proof of Theorem 5.1 in [10]. We outline it only. In the following, \( E \) is an r.i. function space on \((0, \infty)\) with the RNP.

We use the following characterization of the RNP due to Bukhvalov and Danilevich [1]. Let \( X \) be a Banach space. Let \( h_\infty(X) \) denote the space of all bounded harmonic \( X \)-valued functions in the unit disc of the complex plane. Then \( X \) has the RNP iff every function \( f \in h_\infty(X) \) admits almost everywhere radial limits in \( X \) on the unit circle \( T \), that is, \( \lim_{r \to 1} f(re^{i\theta}) \) exists in \( X \) almost everywhere on \( T \).

Let \( f \in h_\infty(L_E(M, \tau)) \). We show that \( f \) admit almost everywhere radial limits in \( L_E(M, \tau) \) on \( T \), from which the theorem follows. Write

\[
f(re^{i\theta}) = \sum_{n \in \mathbb{Z}} a_n r^{|n|} e^{in\theta}, \quad 0 \leq r < 1, \ 0 \leq \theta \leq 2\pi,
\]

where \( a_n \in L_E(M, \tau) \ (n \in \mathbb{Z}) \) and \( \limsup_{n \to \pm\infty} \|a_n\|_E^{1/n} < 1 \). By the semifiniteness of \( \tau \) we find an orthogonal family \( \{e_i\}_{i \in I} \) of projections in \( M \) such that \( \tau(e_i) < \infty \) for every \( i \in I \) and such that

\[
1 = \sum_{i \in I} e_i \quad (\text{convergence in the strong operator topology}).
\]

By [10, Lemma 5.7], for every \( n \in \mathbb{Z} \), \( \{i \in I : \|a_n e_i \|_E \neq 0 \text{ or } \|a_n e_i \|_E \neq 0 \} \) is at most countable; so that there exists an at most countable subset \( \{e_k\}_{k \geq 0} \) of \( \{e_i\}_{i \in I} \) such that \( \|a_n e_k \|_E \neq 0 \) or \( \|a_n e_k \|_E \neq 0 \) for some \( n \in \mathbb{Z} \). Let \( e = \sum_{k \geq 0} e_k \). Then we have \( ef(z) = f(z)e = f(z) \) (\( z \in D \)). Therefore replacing \( M \) by \( eMe \) and \( \tau \) by its restriction to \( eMe \), we can assume \( e = 1 \). For \( j \geq 0 \), \( k \geq 0 \), set \( e_{jk} = e_j \vee e_k \) (maximum taken in the lattice of all the
projections in \( M \). Now let \( M_{e_{jk}} = e_{jk}M e_{jk} \) and \( \tau_{e_{jk}} \) be the restriction of \( \tau \) to \( M_{e_{jk}} \) \((j \geq 0, k \geq 0)\). \( \tau_{e_{jk}} \) is a finite trace on \( M_{e_{jk}} \). For \( j \geq 0, k \geq 0 \) consider \( f_{jk}(z) = e_{jk} f(z) e_{jk} \) \((z \in D)\). Regarded as a function with values in \( L_E(M_{e_{jk}}, \tau_{e_{jk}}) \), \( f_{jk} \in h_\infty(L_E(M_{e_{jk}}, \tau_{e_{jk}})) \) and
\[
\| f_{jk}(z) \|_{L_E(M_{e_{jk}}, \tau_{e_{jk}})} \leq \| f(z) \|_{L_E(M, \tau)}, \quad z \in D.
\]
By Lemma 3, \( L_E(M_{e_{jk}}, \tau_{e_{jk}}) \) has the RNP; so in \( L_E(M_{e_{jk}}, \tau_{e_{jk}}) \)
\[
\lim_{r \to 1} f_{jk}(r e^{i\theta}) = \varphi_{jk}(e^{i\theta}) \quad \text{almost everywhere on } T.
\]
We can evidently extend the boundary function \( \varphi_{jk} \) to a function with values in \( L_E(M, \tau) \). This new function is still denoted by \( \varphi_{jk} \). Then it satisfies
\[
e_{jk} \varphi_{jk}(e^{i\theta}) = \varphi_{jk}(e^{i\theta}) e_{jk} = \varphi_{jk}(e^{i\theta}) \quad \text{almost everywhere on } T;
\]
furthermore, the above almost everywhere radial limits also exist in \( L_E(M, \tau) \). Then by [10, Lemma 5.7], we can show that \( \sum_{j \geq 0} \sum_{k \geq 0} \varphi_{jk}(e^{i\theta}) \) converges in \( L_E(M, \tau) \) almost everywhere on \( T \) (cf. [10] for more details). Let
\[
\varphi(e^{i\theta}) = \sum_{j \geq 0} \sum_{k \geq 0} \varphi_{jk}(e^{i\theta}), \quad 0 \leq \theta \leq 2\pi.
\]
Then \( \varphi \) is a bounded measurable function on \( T \) with values in \( L_E(M, \tau) \) since for any \( m \geq 0, n \geq 0 \)
\[
\operatorname{ess} \sup_{\theta \in T} \left\| \sum_{j=0}^{m} \sum_{k=0}^{n} \varphi_{jk}(e^{i\theta}) \right\|_{L_E(M, \tau)} \leq \sup_{z \in D} \left\| \sum_{j=0}^{m} \sum_{k=0}^{n} f_{jk}(z) \right\|_{L_E(M, \tau)} \leq \sup_{z \in D} \| f(z) \|_{L_E(M, \tau)}.
\]
Let
\[
F(r e^{i\theta}) = \int_{0}^{2\pi} \varphi(e^{i\eta}) P_r(\theta - \eta) \frac{d\eta}{2\pi} \quad (0 \leq r < 1, \ 0 \leq \theta \leq 2\pi)
\]
be the Poisson integral of \( \varphi \) in the unit disc, \( P_r \) being the Poisson kernel. By the dominated convergence theorem
\[
F(z) = \sum_{j \geq 0} \sum_{k \geq 0} F_{jk}(z), \quad z \in D,
\]
where \( F_{jk} \) is the Poisson integral of \( \varphi_{jk} \) \((j \geq 0, k \geq 0)\). Since \( \varphi_{jk} \) is the almost everywhere radial limit of \( f_{jk} \), \( F_{jk} = f_{jk} \); so that
\[
F(z) = \sum_{j \geq 0} \sum_{k \geq 0} f_{jk}(z), \quad z \in D.
\]
On the other hand, it is clear that
\[
f(z) = \sum_{j \geq 0} \sum_{k \geq 0} f_{jk}(z), \quad z \in D.
\]
Consequently, $F = f$. Hence $f$ is the Poisson integral of $\varphi$. Therefore
\[
\lim_{r \to 1} f(re^{i\theta}) = \varphi(e^{i\theta}) \quad \text{in } L_E(M, \tau)
\]
almost everywhere on $T$, which completes the proof of the theorem.

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**References**


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