AN IN Variant ON 3-DIMENSIONAL LIE ALGEBRAS

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Abstract. We construct an extra symmetric bilinear form on a 3-dimensional Lie algebra \( g \) which induces an invariant \( \chi(g) \) on \( g \). Moreover it provides a new viewpoint for the classical classification of 3-dimensional Lie algebras.

In this note, we shall construct an extra symmetric bilinear form \( S \) on a 3-dimensional Lie algebra, which provides new viewpoints for the classical classification of 3-dimensional Lie algebras.

Let \( g \) be a 3-dimensional Lie algebra with the Lie bracket \([ , ]\) over a field \( k \) of characteristic \( \neq 2 \). Let \( \{e_1, e_2, e_3\} \) be a fixed basis. There is a canonical identification \( \wedge^2 g^* \cong g \) by \( e_1 = e_2^* \wedge e_3^* \), \( e_2 = e_3^* \wedge e_1^* \), and \( e_3 = e_1^* \wedge e_2^* \), where \( \{e_1^*, e_2^*, e_3^*\} \) is the dual basis of \( g^* \) with respect to \( \{e_1, e_2, e_3\} \). Then the bracket \([ , ]\) \( \in (\wedge^2 g^*) \otimes g \) is considered as an element of \( g^* \otimes g \) by the identification and induces a bilinear form \( L : g^* \otimes g \rightarrow k \), which is invariant under the change of a basis up to scalar multiplications in \( k \). If we set the structure constants of the bracket \([ , ]\) by

\[
[e_2, e_3] = a_{11}e_1 + a_{12}e_2 + a_{13}e_3, \\
[e_3, e_1] = a_{21}e_1 + a_{22}e_2 + a_{23}e_3, \\
[e_1, e_2] = a_{31}e_1 + a_{32}e_2 + a_{33}e_3,
\]

(1)

then the representation matrix of \( L \) with respect to the basis \( \{e_1^*, e_2^*, e_3^*\} \) is written by \( A = (a_{ij})_{i,j=1,2,3} \). If we change the basis on \( g \) by a matrix \( P = (p_{ij})_{i,j=1,2,3} \in \text{GL}(3, k) \) such that \( e_i' = \sum_{j=1}^{3} p_{ij} e_j \), then new structure constants \( A' \) are given by \( A' = (\det P)P^{-1}A^tP^{-1} \).

Now, we define another bilinear form \( S : g \times g \rightarrow k \) by

\[
S(u, v) = L(u_1^*, v_1^*)L(u_2^*, v_2^*) - L(u_1^*, v_2^*)L(u_2^*, v_1^*) \quad \text{for} \ u, v \in g,
\]

(2)

where \( u = u_1^* \wedge u_2^* \) and \( v = v_1^* \wedge v_2^* \) with respect to the identification \( \wedge^2 g^* \cong g \). Then it can be easily checked that the representation matrix of \( S \) coincides with the cofactor matrix \( A^* \) of \( A \). Since \( A'^* = P^tA^*P \), the bilinear form \( S \) is determined independently of the choice of a basis. The following lemma is immediately obtained from the Jacobi identity.

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Lemma. The bilinear form $S$ is symmetric, namely, $A^* = A^*$.

It should be remarked that $g$ is unimodular if and only if the matrix $A$ is symmetric. When $k$ is algebraically closed, the isomorphism classes of 3-dimensional unimodular Lie algebras are classified by the rank of the matrix $A$.

Theorem 1. Let $g$ be a 3-dimensional Lie algebra. Then the bilinear form $S$ defined by (2) is proportional to the Killing form $F$ of $g$.

Proof. Let $B$ be a representation matrix of the Killing form $F$. By a straightforward calculation, one can obtain the identity $B = A^* - 2A^*$, where $A^*$ is the cofactor matrix of $A = A - A^*$. If $g$ is unimodular, then $A^* = 0$ and $F = -2S$ holds. So we may assume that $g$ is not unimodular. Then $g$ is solvable and the basis $\{e_1, e_2, e_3\}$ can be chosen such that $a_{ij} = a_{ji} = 0 (i = 1, 2, 3)$ (see [1, p. 12]). Then one can easily verify that $A^* - 2A^*$ is proportional to $A^*$. This proves the theorem.

By the theorem, we can define an invariant $\chi(g) \in \mathbb{P} = k \cup (\infty)$ by $F = (\chi(g) - 2)S$, unless $F = S = 0$; namely, $g$ is neither Heisenberg nor abelian. There is another exceptional Lie algebra denoted by $t$, which is characterized by the property that the matrix $A$ is skew symmetric. One can easily verify that the well-known classification theorem (e.g., [1, p. 13; 2, Lemma 4.10]) of 3-dimensional Lie algebras is rewritten in the following

Theorem 2. Let $g$ be a 3-dimensional Lie algebra that is neither unimodular nor isomorphic to $t$. Then there exists a basis $\{e_1, e_2, e_3\}$ of $g$ such that

$$[e_3, e_2] = e_1, \quad [e_2, e_1] = -e_1 + \frac{1}{\chi(g)}e_2 \quad \text{and} \quad [e_1, e_2] = 0.$$

References


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