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FURUTA'S INEQUALITY AND A GENERALIZATION OF ANDO'S THEOREM

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Dedicated to Professor Tsuyoshi Ando on his sixtieth birthday.

ABSTRACT. As a continuation of our preceding notes, we discuss Furuta's inequality under the chaotic order defined by $\log A \geq \log B$ for positive invertible operators A and B. We prove that Furuta's type inequalities $(B^r A^p B^r)^{2r/(p+2r)} \geq B^{2r}$ and $A^{2r} \geq (A^r B^p A^r)^{2r/(p+2r)}$ hold true for $p \geq r \geq 0$, which is a generalization of an inequality due to Ando (Math. Ann. **279** (1987), 157–159).

1. INTRODUCTION

In [6] Furuta established an operator inequality that is an extension of the Lowner-Heinz inequality.

Furuta's inequality. *Let A and B be positive operators acting on a Hilbert space. If $A \geq B \geq 0$ then*

$$(B^r A^p B^r)^{(1+2r)/(p+2r)} \geq B^{(1+2r)} \tag{1}$$

and

$$A^{(1+2r)} \geq (A^r B^p A^r)^{(1+2r)/(p+2r)} \tag{2}$$

for all $p \geq 1$ and $r \geq 0$.

In succesion, Ando [1] proved an operator monotonelike property for the exponential function. Inspired by this and the exponential order due to Hansen [9], we introduced the chaotic order $A \gg B$ by $\log A \geq \log B$ among positive invertible operators. It is just opposite to the exponential one. Thus, Ando's result in [1] is rephrased that $A \gg B$ if and only if

$$A^p \geq (A^{p/2} B^p A^{p/2})^{1/2} \tag{3}$$

for all $p \geq 0$. Obviously, (3) is equivalent to $B^p \leq (B^{p/2} A^p B^{p/2})^{1/2}$.

In the preceding paper [5] we proved an operator inequality like Furuta's one.

Theorem A. *Let A and B be positive invertible operators. If $A \gg B$ then*

$$B^r A^p B^r \gg B^{p+2r} \tag{4}$$

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and

$$A^{p+2r} \gg A^r B^p A^r \tag{5}$$

for all $p, r \geq 0$.

On the other hand, Furuta [8] recently established further developments of his inequality by considering some operator functions with monotone property, cf. also [3].

In this note, we observe the monotonity of an operator function $H(s) = (AB^sA)^{2/(s+2)}$ under the assumption $A \gg B$ and give a simple proof to Theorem A. Furthermore, we extend Ando's inequality (3) to an inequality of a type of Furuta's one. More precisely, we prove that if $A \gg B$ then

$$(B^rA^pB^r)^{2r/(p+2r)} \geq B^{2r} \quad \text{and} \quad A^{2r} \geq (A^rB^pA^r)^{2r/(p+2r)}$$

for $p \geq r \geq 0$.

2. Monotonity

As in [4, 5] (cf. [2, 3, 7, 10, 11]) the flame of our discussion is means of operators established by Kubo and Ando [12]. A binary operation m among positive operators is called a mean if m is uppersemicontinuous, preserves (the usual) order, and satisfies the transformer inequality

$$T^*(AmB)T \leq T^*ATmT^*BT$$

for all T. We note that the above inequality is changed into the equality

$$T^*(AmB)T = T^*ATmT^*BT$$

for all invertible T. By the principal result in [12], there is a unique mean m_s corresponding to the operator monotone function x^s for $0 \leq s \leq 1$;

$$1m_sx = x^s$$

for $t \geq 0$. In order to follow the argument of [11], we use Ando's result (3) and the Lowner-Heinz inequality, simply (LH); $A \geq B \geq 0$ implies $A^s \geq B^s$ for $0 \leq s \leq 1$.

Lemma 1. *If* $A \gg B$ *then*

$$(B^{n/2}A^2B^{n/2})^{1/(n+2)} \geq B$$

for all positive integers n.

Proof. First of all, we prove the case $n = 1$. If we put $C = B^{1/2}AB^{1/2}$, then it follows from (3) that $C^{1/2} \geq B$; therefore, we have

$$B^{1/2}A^2B^{1/2} = CB^{-1}C \geq CC^{-1/2}C = C^{3/2},$$

so (LH) implies

$$(B^{1/2}A^2B^{1/2})^{1/3} \geq C^{1/2} \geq B.$$

Next we suppose that the statement holds true for n, i.e., $D^{1/(n+2)} \geq B$, where $D = B^{n/2}A^2B^{n/2}$. Then, for $m = m_{1/(n+3)}$,

$$\begin{aligned}(B^{(n+1)/2}A^2B^{(n+1)/2})^{1/(n+3)} &= B^{1/2}(B^{-1}mD)B^{1/2} \\ &\geq B^{1/2}(D^{-1/(n+2)}mD)B^{1/2} \\ &= B^{1/2}IB^{1/2} = B,\end{aligned}$$

which shows that it is true for $n + 1$.

Corollary 2. *If* $A \gg B$ *then*

$$(B^{s/2}A^2B^{s/2})^{1/(s+2)} \geq B$$

for all $s \geq 1$.

Proof. As in the proof of [11, Corollary 2], we take an integer n with $n \leq s < n+1$. If D is as above and $m = m_{1/(s+1)}$, then it follows from Lemma 1 and (LH) that

$$\begin{aligned} B^{-s}mA^2 &= B^{-n/2}(B^{n-s}mD)B^{-n/2} \geq B^{-n/2}(D^{(n-s)/(n+2)}mD)B^{-n/2} \\ &= B^{-n/2}D^{(n+1-s)/(n+2)}B^{-n/2} \\ &\geq B^{-n/2}B^{n+1-s}B^{-n/2} = B^{1-s}. \end{aligned}$$

By multiplying $B^{s/2}$ on both sides, we have the conclusion.

Now we observe an operator function with monotone property.

Lemma 3. *If* $A \gg B$ *then*

$$AB^{s+t}A \leq (AB^sA)^{(s+t+2)/(s+2)}$$

for all $s \geq 1$ *and* $1 \geq t > 0$.

Proof. If we put $E = B^{s/2}A^2B^{s/2}$ and $m = m_{t/(s+2)}$, then $E^{t/(s+2)} \geq B^t$ by Corollary 2 and (LH). Hence, we have

$$\begin{aligned} AB^{s+t}A &= AB^{s/2}B^tB^{s/2}A \\ &\leq AB^{s/2}E^{t/(s+2)}B^{s/2}A = AB^{s/2}(1mE)B^{s/2}A \\ &= AB^sAmAB^sA^2B^sA = (AB^sA)^{1+t/(s+2)}. \end{aligned}$$

The following result might be a variant of Furuta's theorem in [8].

Theorem 4. *If* $A \gg B$ *then*

$$H(s) = (AB^sA)^{2/(s+2)}$$

is monotone decreasing (*on usual order*) *for* $s \geq 1$.

Proof. For $s \geq 1$ and $1 \geq t > 0$, since $0 < 2/(s+t+2) < 1$, it follows from Lemma 3 and (LH).

Corollary 5. *If* $A \gg B$ *then*

$$K(n) = (AB^nA)^{1/(n+2)}$$

is monotone decreasing for nonnegative integer n.

Proof. We only have to show $K(0) \geq K(1)$. If we put $C = A^{1/2}BA^{1/2}$ and $m = m_{1/3}$, then it follows from (3) that

$$K(1) = A^{1/2}(A^{-1}mC)A^{1/2} \leq A^{1/2}(C^{-1/2}mC)A^{1/2} = A = K(0).$$

3. Furuta's inequality

First of all, we give a proof to Theorem A based on results in the preceding section.

Proof of Theorem A. Assume that $r \le p$. Since $s = p/r \ge 1$ and $A^r \gg B^r$, Theorem 4 and Corollary 5 imply that

$$(A^r(B^r)^s A^r)^{1/(s+2)} \le A^r,$$

and so

$$(A^r B^p A^r)^{r/(p+2r)} \le A^r.$$

Then we have the conclusion by taking the logarithm.

Next assume that $r \ge p$. Noting that

$$A^r B^p A^r = (A^{p/2})^{2r/p}(B^{p/2})^2(A^{p/2})^{2r/p}$$

and $2r/p \ge 1$, it follows from Corollary 2 on $s = 4r/p$ that

$$(A^r B^p A^r)^{p/(4r+2p)} \le A^{p/2},$$

which implies the conclusion.

Now we recall Ando's inequality (3), which is expressed as

$$A^{-p} m_{1/2} B^p \le I \tag{3'}$$

for all $p \ge 0$ if $A \gg B$. Finally, we consider a generalization to an inequality of a type of Furuta's,

$$A^{-2r} m_{2r/(p+2r)} B^p \le I. \tag{6}$$

Clearly we have $(3')$ for $p = 2r$. The following theorem says that (6) is valid under a certain restriction, and it is also an extension of our result in [4].

Theorem 6. *If $A \gg B$ then*

$$(B^r A^p B^r)^{2r/(p+2r)} \ge B^{2r}$$

and

$$A^{2r} \ge (A^r B^p A^r)^{2r/(p+2r)}$$

for $p \ge r \ge 0$.

Proof. . First of all, we point out that $(ABA)^{2/3} \le A^2$. As a matter of fact, if we put $m = m_{2/3}$ and $C = A^{1/2}BA^{1/2}$, then (3) implies $C^{1/2} \le A$, and so

$$\begin{aligned}(ABA)^{2/3} &= A^{1/2}(A^{-1}mC)A^{1/2} \le A^{1/2}(C^{-1/2}mC)A^{1/2} \\ &= A^{1/2}C^{1/2}A^{1/2} \le A^{1/2}AA^{1/2} = A^2.\end{aligned}$$

Combining this and Theorem 4, we have $H(s) \le A^2$ for all $s \ge 1$, where $H(s)$ is as in Theorem 4. Hence, if we put $s = p/r \le 1$, then it implies

$$(A^r B^p A^r)^{2p/(p+2r)} = (A^r(B^r)^s A^r)^{2/(s+2)} \le (A^r)^2 = A^{2r}.$$

If we see the above theorem as (6) holds true for $p \ge r \ge 0$, then we have a generalization of Ando's theorem in [1]:

Theorem 7. *For positive invertible operators A and B, $A \gg B$ if and only if*

$$A^{-2r} m_{2r/(p+2r)} B^p \le I$$

holds true for $p \ge r \ge 0$.

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