A FUNCTIONAL ANALYSIS PROOF
OF THE EXISTENCE OF HAAR MEASURE
ON LOCALLY COMPACT ABELIAN GROUPS

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(Communicated by Andrew M. Bruckner)

Abstract. A simple proof of the existence of Haar measure on locally compact abelian groups is given. The proof uses the Markov-Kakutani fixed-point theorem.

It is very well known that every locally compact group has a Haar measure and that the Haar measure is unique up to a positive multiplicative constant. Several different proofs have been given, all of them somewhat difficult. (See [N] for two proofs as well as references to others). In most of these proofs, the existence and uniqueness of Haar measure are established separately. For compact groups, a simple proof of the existence and uniqueness of Haar measure was given by von Neumann [vN], and his proof can be made even simpler by using the Kakutani fixed-point theorem (see [R2]). For locally compact abelian groups, uniqueness of Haar measure is easily established (see [R1, p. 2]). The purpose of this short note is to present a simple proof of the existence of Haar measure for these groups. The proof will make use of the Markov-Kakutani fixed-point theorem, which we recall below. It is known that this fixed-point theorem can be used to prove that every locally compact abelian group has an invariant mean (see [P, p. 113]). For compact groups an invariant mean and a Haar measure are the same thing, but for noncompact groups this is obviously not the case.

Theorem (Markov-Kakutani). Let \( K \) be a nonempty compact convex subset of a (Hausdorff) topological vector space. Let \( \mathcal{F} \) be a commuting family of continuous affine mappings of \( K \) into itself. Then there exists a point \( p \in K \) such that \( Tp = p \) for all \( T \in \mathcal{F} \).

A proof can be found in [C, pp. 155–156]. (There the theorem is stated only for locally convex spaces, but local convexity is not needed in the proof.)

We will also need two lemmas.
Lemma 1. Suppose $G$ is a topological group and $N$ is a neighborhood of the identity in $G$ that is symmetric (i.e., $N^{-1} = N$). Then there exists a subset $S$ of $G$ such that for each $g$ in $G$ the set $gN \cdot N$ contains at least one element of $S$ and the set $gN$ contains at most one element of $S$.

Proof. Let $\mathcal{F}$ be the collection of all subsets $T$ of $G$ such that

\[ p^{-1}q \not\in N \cdot N \quad \text{for all } p, q \in T. \]

By applying Zorn's lemma, we see that $\mathcal{F}$ has a maximal element $S$. Now if $g \in G$, then there is some $s$ in $S$ such that $g^{-1}s \in N \cdot N$, for otherwise the set $S \cup \{g\}$ would be a member of $\mathcal{F}$ strictly containing $S$. Moreover, if there were two distinct points $s_1, s_2$ in $S$ such that both $g^{-1}s_1$ and $g^{-1}s_2$ were in $N$, then we would have $s_1^{-1}s_2 = s_1^{-1}gg^{-1}s_2 \in N^{-1} \cdot N = N \cdot N$, a contradiction. Thus, there is at most one $s$ in $S$ such that $g^{-1}s \in N$. 

Lemma 2. Let $X$ be a vector space, and let $X^*$ denote the space of all linear functionals on $X$ with the weak*-topology (i.e., the weak topology induced by $X$). If $K$ is a closed subset of $X^*$ such that for each $x \in X$ the set $\{Ax : A \in K\}$ is bounded, then $K$ is compact.

The proof of this lemma is very similar to the proof of the Banach-Alaoglu theorem and is essentially contained in [DS, pp. 423-424]. A more succinct statement of the conclusion is that every closed bounded set in $X^*$ is compact.

Proof of the existence of Haar measure on locally compact abelian groups. Let $G$ be a locally compact abelian group. Let $C_c(G)$ denote the space of compactly supported continuous functions on $G$, and let $C_c(G)^*$ denote the space of all linear functionals on $C_c(G)$ with the weak*-topology (i.e., the weak topology induced by $C_c(G)$). If $f \in C_c(G)$ and $a \in G$, then $f_a$ (the translate of $f$ by $a$) is defined by $f_a(x) = f(a+x)$. For each $a$ in $G$, define $T_a : C_c(G)^* \to C_c(G)^*$ by the equation

\[ (T_a\Lambda)(f) = \Lambda(f_a) \quad (\Lambda \in C_c(G)^*, \; f \in C_c(G)). \]

Then each $T_a$ is a continuous linear operator. To establish the existence of Haar measure on $G$ we must simply show that there is a nonzero positive linear functional on $C_c(G)$ that is fixed by every $T_a$.

Fix a symmetric neighborhood $N$ of the identity in $G$ with compact closure. Let $K$ be the set of positive linear functionals $\Lambda$ on $C_c(G)$ that satisfy the following two conditions:

(i) $\Lambda(f) \leq 1$ whenever $f$ is a nonnegative function in $C_c(G)$ that is bounded above by 1 and whose support is contained in $a + N$ for some $a \in G$, and

(ii) $\Lambda(f) \geq 1$ whenever $f$ is a nonnegative function in $C_c(G)$ that is equal to 1 on $a + N + N$ for some $a \in G$.

Then $K$ is clearly closed and convex in $C_c(G)^*$. Moreover, by a partition of unity argument every nonnegative function in $C_c(G)$ can be written as a finite sum of nonnegative continuous functions each of which has support in $a + N$ for some $a \in G$. It follows that condition (i) in the definition of $K$ implies that for each function $f$ in $C_c(G)$, the set $\{\Lambda(f) : \Lambda \in K\}$ is bounded. Therefore by Lemma 2, $K$ is compact.
Let $S$ be as in Lemma 1, and note that the functional that consists of a point mass at each point of $S$ (i.e., the functional $f \mapsto \sum_{s \in S} f(s)$) is in $K$. Thus $K$ is nonempty.

It is clear from the definition of $K$ that each of the operators $T_a$ maps $K$ into itself. Hence, since the operators $T_a$ ($a \in G$) form a commuting family, the Markov-Kakutani fixed-point theorem shows that they have a common fixed-point in $K$. Since all the elements of $K$ are nonzero positive linear functionals on $C_c(G)$, the proof is complete. □

References