REDUCTION OF PREQUANTIZED $Mp^c$ STRUCTURES

P. L. ROBINSON

(Communicated by Jonathan M. Rosenberg)

INTRODUCTION

The Marsden-Weinstein reduction procedure fashions new symplectic manifolds from Hamiltonian actions on old symplectic manifolds. It is of considerable interest to determine the way in which quantization of the reduced phase spaces relates to that of the original phase spaces. In particular, it is natural to ask for conditions under which prequantization data may be passed through the operation of reduction. Note that reduced phase spaces can fail to be metaplectic, so in general it cannot be expected that traditional prequantization data (Kostant-Souriau line bundle plus metaplectic structure: see [2, 3] for example) will pass to reduced phase spaces; moreover, even when reduced phase spaces admit metaplectic structures, these do not generally arise from metaplectic structures on the original phase spaces in a natural manner. Following Hess, we take the view that prequantization data consists of a prequantized $Mp^c$ structure: see [6] for full details. In this note, we describe when and how such prequantization data may be passed to reduced phase spaces. En route, we encounter a generalized version of the Bohr-Wilson-Sommerfeld rule as an obstruction to this passage. We illustrate our account by reference to coadjoint orbits as reduced phase spaces.

1. THE GENERAL CONSTRUCTION

Let $(Z, \Omega)$ be a symplectic manifold equipped with a Hamiltonian action of the Lie group $G$, and suppose the moment map $J : Z \to \mathfrak{g}^*$ to be equivariant with respect to the given left action of $G$ on $Z$ and the coadjoint action of $G$ on $\mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$ be a regular value of $J$, let $S = J^{-1}(\mu)$ be the preimage of $\mu$ in $Z$ under $J$, and let $H = G_\mu$ be the stabilizer of $\mu$ in $G$. Under a variety of assumptions, the orbit space $H\backslash S =: M$ is a manifold and the projection $S \to M$ a submersion; for example, it is enough to assume that the $H$-action on $S$ is proper and free. We make such an assumption, in which case $M$ acquires a canonical symplectic form $\omega$. The symplectic manifold $(M, \omega)$ is called the Marsden-Weinstein reduction (or reduced phase space) of $(Z, \Omega)$.  

Received by the editors January 29, 1990 and, in revised form, December 27, 1990.

1980 Mathematics Subject Classification (1985 Revision). Primary 58F05, 58F06; Secondary 53C57.

The author acknowledges partial financial support from the National Science Foundation.
at $\mu$. If $H^0$ is the identity component of $H$ then $H^0 \backslash S \cong M'$ also carries a canonical symplectic form $\omega'$ and the symplectic manifold $(M', \omega')$ naturally covers $(M, \omega)$. For an account of Marsden-Weinstein reduction, we refer to [1]. For our purposes, we need further information on the symplectic geometry of reduced phase spaces.

Fix $x \in S$ and write $m = [x] \in M$ for the $H$-orbit of $x$ in $S$. The tangent space to the $G$-orbit of $x$ at $x$ is precisely the symplectic orthogonal of $T_xS$ in $T_xZ$:

$$T_x(G \cdot x) = (T_xS)^\perp;$$

also,

$$T_x(H \cdot x) = (T_xS) \cap T_x(G \cdot x).$$

Consequently,

$$T_x(H \cdot x) = (T_xS) \cap (T_xS)^\perp.$$

Moreover, the realization of $M$ as the orbit space $H \backslash S$ gives us a canonical short exact sequence

$$0 \to T_x(H \cdot x) \to T_xS \to T_mM \to 0.$$

If we define

$$E_x := (T_xS)/[(T_xS) \cap (T_xS)^\perp]$$

then it follows that we have a canonical isomorphism from $E_x$ to $T_mM$; this isomorphism pulls back $\omega_m$ to the symplectic form on $E_x$ induced from $\Omega_x$ on $T_xZ$. Suppose also that $y \in S$ with $m = [y]$. In this case, $y = h \cdot x$ for some $h \in H$; differentiating the action of $h$ gives an isomorphism $h_*$ from $T_xS$ to $T_yS$ carrying $(T_xS)^\perp$ to $(T_yS)^\perp$. As a result, there is induced a symplectic isomorphism $h_* : E_x \to E_y$. Under a variety of assumptions, the isomorphism $h_*$ is independent of the choice of $h$ such that $y = h \cdot x$; for example, if the $H$-action on $S$ is free then $h$ is unique. We make such an assumption, in which case it is now clear that the canonical $H$-action (by $#$) on the symplectic vector bundle $E$ over $S$ is such that the quotient $H \backslash E$ is canonically $TM$ (as a symplectic vector bundle) just as the quotient of $S$ by $H$ is $M$.

At the level of symplectic frame bundles, a similar picture emerges. Model $(Z, \Omega)$ on the symplectic vector space $(V, \Omega_0)$: thus, the fibre of the symplectic frame bundle $Sp(Z, \Omega)$ over $z \in Z$ consists of all symplectic linear isomorphisms $V \to T_zZ$. If we model $S \subset Z$ on the subspace $W \subset V$ then the bundle $Sp(Z, \Omega; S)$ of adapted frames has as fibre over $x \in S$ the set of all $b \in Sp(Z, \Omega)_x$ such that $b(W) = T_xS$; the structure group of $Sp(Z, \Omega; S)$ is $Sp(V; W')$—the subgroup of the full symplectic group $Sp(V)$ consisting of all $g$ with $g(W) = W$. There is a canonical homomorphism from $Sp(V; W)$ to $Sp(W/(W \cap W^\perp))$; the bundle associated to $Sp(Z, \Omega; S)$ via this homomorphism is naturally the symplectic frame bundle $Sp(E)$ when we model $E$ on the symplectic vector space $W/(W \cap W^\perp)$. Now the symplectic action of $G$ on $(Z, \Omega)$ lifts naturally to an action of $G$ on $Sp(Z, \Omega)$: explicitly, if $g \in G$ and $b \in Sp(Z, \Omega)$ then $g \cdot b = g_* \circ b$. The resulting action of $H$ maps $Sp(Z, \Omega; S)$ to itself and induces an action of $H$ on $Sp(E)$; the quotient of $Sp(E)$ by $H$ is canonically $Sp(M, \omega)$.
We remark that obvious changes take place if we factor by the action of $H^0$ rather than by the action of $H$. Thus: the quotient of $E$ by $H^0$ is $TM'$ and the quotient of $Sp(E)$ by $H^0$ is $Sp(M', \omega')$.

In order to introduce prequantization data, it is convenient to interpose some remarks of a group theoretical nature; full details appear in [6]. Recall that if $V$ is a symplectic vector space with symplectic form $\Omega_0$ then the symplectic group $Sp(V)$ consists of all the linear automorphisms $g$ of $V$ with $g^*\Omega_0 = \Omega_0$. Fix a positive scalar $h = 2\pi \hbar$ and an irreducible projective unitary representation $\rho$ of $V$ on a Hilbert space $H$ satisfying

$$\rho(x)\rho(y) = \exp \left\{ \frac{1}{2i\hbar} \Omega_0(x, y) \right\} \rho(x + y)$$

whenever $x, y \in V$. As a consequence of the uniqueness theorem due to Stone and von Neumann, if $g \in Sp(V)$ then $\rho \circ g$ is unitarily equivalent to $\rho$: there exists a unitary operator $U$ on $H$ such that

$$v \in V \Rightarrow \rho(gv) = U\rho(v)U^{-1};$$

irreducibility of $\rho$ guarantees uniqueness of $U$ up to scalar multiples. $Mp^c(V)$ denotes the group of all such unitaries $U$ on $H$ as $g$ ranges over $Sp(V)$; thus, we have a short exact sequence

$$1 \to U(1) \to Mp^c(V) \xrightarrow{\sigma} Sp(V) \to 1$$

where $\sigma$ sends $U$ to $g$. The group $Mp^c(V)$ has a unique unitary character $\eta$ such that $\eta(\lambda) = \lambda^2$ when $\lambda \in U(1)$. Although the exact sequence for $Mp^c(V)$ does not split, the derived exact sequence of Lie algebras

$$0 \to u(1) \to mp^c(V) \xrightarrow{\sigma} sp(V) \to 0$$

is canonically split by $\frac{1}{2} \eta_* : mp^c(V) \to u(1)$. We remark that the kernel of $\eta$ is a connected double cover of $Sp(V)$ known as the metaplectic group.

We are now in a position to introduce prequantization data for the symplectic manifold $(Z, \Omega)$. This data comes in two parts: an $Mp^c(V)$ structure (which makes sense for any symplectic vector bundle) with a prequantum form (which essentially assumes a symplectic manifold). For convenience, we briefly describe these notions here; for full details, see [6].

An $Mp^c$ structure for the symplectic vector bundle $B$ over the manifold $X$ is a principal $Mp^c(V)$ bundle $P$ on $X$ together with a $\sigma$-equivariant bundle map from $P$ to the symplectic frame bundle $Sp(B)$. In fact, every symplectic vector bundle $B \to X$ admits $Mp^c$ structures—there is no obstruction to their existence; further, up to the natural notion of equivalence, the $Mp^c$ structures for $B \to X$ are parametrized by $H^2(X; Z)$. We remark that, in contrast, $B \to X$ admits the more familiar metaplectic structures precisely when the Stiefel-Whitney class $w_2(B)$ vanishes.

A prequantized $Mp^c$ structure $(P, \gamma)$ for $(Z, \Omega)$ is an $Mp^c$ structure $P$ for the symplectic vector bundle $TZ$ provided with a prequantum form $\gamma$: an $Mp^c$-invariant $u(1)$-valued one-form on $P$ such that $\gamma(\tilde{z}) = \frac{1}{2} \eta_* z$ and $d\gamma = (1/i\hbar) \pi^* \Omega$, where $z \in mp^c(V)$ determines the fundamental vector field $\tilde{z}$ on $P$ and where $\pi : P \to Z$ is the bundle projection. We remark that a prequantum form $\gamma$ on $P$ corresponds naturally with a connexion $\alpha^\gamma$ of curvature $(2/i\hbar)\Omega$ in the principal $U(1)$ bundle $P(\eta)$ associated to $P$ via the
unitary character $\eta$. Whereas $Mp^c$ structures themselves always exist, $(Z, \Omega)$ admits prequantized $Mp^c$ structures iff the real cohomology class $[\frac{c_1}{2}] - \frac{1}{2}c(\Omega)^R$ is integral, where $c(\Omega)^R$ is the real first Chern class determined by a unitary reduction of $TZ$; further, when they exist, the inequivalent prequantized $Mp^c$ structures for $(Z, \Omega)$ are parametrized by $H^1(Z; U(1))$.

We can address the problem of passing prequantized $Mp^c$ structures from $(Z, \Omega)$ to its reduced phase space once we have dealt with a further property of $Mp^c$ groups. Let $Mp^c(V; W)$ be the preimage of $Sp(V; W)$ in $Mp^c(V)$ and note that $W/(W \cap W^\perp)$ is a distinguished symplectic subspace of $(W + W^\perp)/(W \cap W^\perp)$. In [6] we show that the canonical homomorphism $\nu$ from $Sp(V; W)$ to $Sp((W + W^\perp)/(W \cap W^\perp))$ lifts naturally to a homomorphism $\tilde{\nu}$ from $Mp^c(V; W)$ to $Mp^c((W + W^\perp)/(W \cap W^\perp))$; this clearly induces a homomorphism from $Mp^c(V; W)$ to (a copy of) $Mp^c((W/(W \cap W^\perp))$, which we also call $\tilde{\nu}$. We remark that these lifted homomorphisms do not generally exist at the level of metaplectic covers.

Now, let $(Q, \delta)$ be a prequantized $Mp^c$ structure for the symplectic manifold $(Z, \Omega)$. The portion $Q^S$ of $Q$ lying over $Sp(Z, \Omega; S)$ is a principal $Mp^c(V; W)$ bundle on $S$; associated to $Q^S$ via the canonical homomorphism $\tilde{\nu}$ is an $Mp^c$ structure $Q_S$ for the symplectic vector bundle $E = TS/[(TS) \cap (TS)^\perp]$ over $S$. The restriction $\delta^S$ of $\delta$ to $Q^S \subset Q$ pushes forward to a form $\delta_S$ on $Q_S$; this is quite easily seen, for example, by regarding the prequantum form $\delta$ on $Q$ as a connexion $\alpha^S$ in the associated principal bundle $Q(\eta)$. It is from $(Q_S, \delta_S)$ that we intend to construct prequantized $Mp^c$ structures for $(M', \omega')$ and $(M, \omega)$.

Recall that $Sp(M', \omega')$ is naturally the quotient of $Sp(E)$ by the canonical action of $H^0$, the identity component of $H$. We would like to produce a prequantized $Mp^c$ structure $(P', \gamma')$ for $(M', \omega')$ as a quotient of $(Q_S, \delta_S)$. In order to do this, we view $\delta_S$ as a connexion in the principal $U(1)$ bundle $Q_S \to Sp(E)$ and require that its holonomy be trivial on each orbit of $H^0$ in $Sp(E)$. When this condition is satisfied, we introduce an equivalence relation on $Q_S$ by declaring that $x, y \in Q_S$ are equivalent if and only if $x$ and $y$ are ends of the $\delta_S$-horizontal lift of some curve in an $H^0$-orbit of $Sp(E)$. The quotient $P'$ of $Q_S$ under this equivalence relation is an $Mp^c$ structure for the reduced phase space $(M', \omega')$ on which $\delta_S$ descends to define a prequantum form $\gamma'$. We remark that the process of reducing a principal $U(1)$ bundle with connexion under a condition of trivial holonomy is well-known: see [7] for example. Note that, since $(Q_S, \delta_S)$ comes from $(Q^S, \delta^S)$ by a push-forward operation, we may replace our earlier holonomy condition by the requirement that $\delta^S$ have trivial holonomy on each orbit of $H^0$ in $Sp(Z, \Omega; S)$.

In terms of our established notation, we have demonstrated the following.

**Theorem 1.** Let $(Q, \delta)$ be a prequantized $Mp^c$ structure for the phase space $(Z, \Omega)$. If the induced connexion $\delta^S$ in the principal bundle $Q^S \to Sp(Z, \Omega; S)$ has trivial holonomy on each $H^0$-orbit in $Sp(Z, \Omega; S)$, then the reduced phase space $(M', \omega')$ naturally acquires a prequantized $Mp^c$ structure $(P', \gamma')$.

The vanishing holonomy condition in this theorem is a generalized version of the (corrected) Bohr-Wilson-Sommerfeld quantization rule. Standard versions of the BWS rule are phrased in terms of polarizations; for example, see [7]. In contrast, our generalized version is polarization-independent: it refers only to
prequantization data, in the form of prequantized $M_p$ structures. Our BWS rule therefore fits quite naturally into the geometric quantization scheme of [6].

Assume the condition of Theorem 1: that $\delta^S$ has trivial holonomy on each $H^0$-orbit in $\text{Sp}(Z, \Omega; S)$. We claim that the action of $H^0$ on $\text{Sp}(Z, \Omega; S)$ lifts naturally to $Q^S$ and that the quotient of $(Q_S, \delta_S)$ by the induced action of $H^0$ is canonically $(P', \gamma')$. Let $q \in Q^S$ and write $b$ for its image in $\text{Sp}(Z, \Omega; S)$. Let $h \in H^0$ and write $h = \exp \xi_n \cdots \exp \xi_1$ for $\xi_1, \ldots, \xi_n$ in the Lie algebra of $H$. For $0 \leq t \leq 1$ and $1 \leq j \leq n$ we put

$$c_j(t) = \exp t \xi_j \cdots \exp \xi_1 \cdot b;$$

the concatenation of the paths $c_1, \ldots, c_n$ is a path $c$ in $\text{Sp}(Z, \Omega; S)$ originating at $b$. We define $h \cdot q$ to be the terminus of the $\delta^S$-horizontal lift of $c$ originating at $q$. The assumption that $\delta^S$ have trivial holonomy on $H^0$-orbits in $\text{Sp}(Z, \Omega; S)$ ensures that we have well-defined an action of $H^0$ on $Q^S$, independently of the way in which elements of $H^0$ are expressed as products. The orbits of $H^0$ in the induced action on $Q_S$ are precisely the points of $P'$. In this way we realize $(P', \gamma')$ as the quotient of $(Q_S, \delta_S)$ under $H^0$.

Retaining the assumption that $\delta^S$ have trivial holonomy on $H^0$-orbits in $\text{Sp}(Z, \Omega; S)$, let us assume further that the induced $H^0$-action on $Q_S$ preserving $\delta_S$ extends to an $H$-action on $(Q_S, \delta_S)$ preserving $\delta_S$. The orbit space $P := H\backslash Q_S$ is an $M_p$ structure for $TM$ on which $\delta_S$ descends to define a prequantum form $\gamma$ by virtue of its invariance. In this way, the reduced phase space $(M, \omega)$ is provided with a prequantized $M_p$ structure. We state this result as follows.

**Theorem 2.** Let $(Q, \delta)$ be a prequantized $M_p$ structure for $(Z, \Omega)$. If $\delta^S$ has trivial holonomy on $H^0$-orbits in $\text{Sp}(Z, \Omega; S)$ and the induced $H^0$-action on $(Q_S, \delta_S)$ extends to an $H$-action on $(Q_S, \delta_S)$ then $(M, \omega)$ naturally acquires a prequantized $M_p$ structure $(P, \gamma)$.

We should point out that our theorems only give sufficient conditions in order that prequantization data pass to reduced phase spaces. We should also point out again that our theorems implicitly assume good behavior of the quotients $H\backslash S$ and $H\backslash E$; for example, we might assume the $H$-action on $S$ to be proper and free.

**2. Coadjoint orbits**

In this section, we apply our general constructions to the particular case of coadjoint orbits. Of course, for this we must realize coadjoint orbits as reduced phase spaces.

Let $G$ be a Lie group and let $Z = T^*G$ be its cotangent bundle with the canonical symplectic form $\omega = d\theta$. The standard left action of $G$ on $(Z, \Omega)$ is Hamiltonian, with equivariant moment map $J: Z \to \mathfrak{g}^*$ given by

$$J(\alpha_g) = \text{Ad}'_g \alpha$$

for $\alpha \in \mathfrak{g}^*$ and $g \in G$; here, $\alpha_g$ denotes the value at $g$ of the left-invariant one-form $\alpha$ and $\text{Ad}'$ denotes the coadjoint representation of $G$ on $\mathfrak{g}^*$. Each $\mu \in \mathfrak{g}^*$ is a regular value of $J$; we have

$$S = J^{-1}(\mu) = \{\alpha_g : \text{Ad}'_g \alpha = \mu\}$$
and
\[ H = G_\mu = \{ h \in G : \text{Ad}'_h \mu = \mu \} . \]

In fact, if \( \beta \) denotes the right-invariant one-form on \( G \) with value \( \mu_e \) at the identity \( e \), then \( \beta : G \to T^* G \) gives a diffeomorphism of \( G \) with \( \beta(G) = S \); moreover, for the standard left actions of \( H \), \( \beta \) is equivariant. The action of \( H \) on \( S \) is certainly proper and free. Further, the reduced phase space \( M = H \backslash S \) may be canonically identified with the coadjoint orbit \( \mathcal{O} = G \cdot \mu \subset \mathfrak{g}^* \); indeed, a canonical identification maps \( H \cdot \beta(g) \in M \) to \( \text{Ad}'_{g^{-1}} \mu \in \mathcal{O} \). Similarly, the reduced phase space \( M' = H^0 \backslash S \) naturally covers the coadjoint orbit \( \mathcal{O} \).

Our aim here is to explicitly construct prequantized \( Mp^c \) structures for \( (M', \omega') \) and \( (M, \omega) \). In order to do so, we must discuss the symplectic frame bundle \( \text{Sp}(Z, \Omega) \) and its subbundle \( \text{Sp}(Z, \Omega; S) \). It transpires that both bundles are canonically trivial, though the canonical trivializations are hardly obvious.

A canonical diffeomorphism \( \Lambda : G \times \mathcal{G}^* \to Z \) is defined by \( \Lambda(g, \alpha) = \alpha_g \) for \( g \in G \) and \( \alpha \in \mathcal{G}^* \). Composing the derivative \( \Lambda_* \) from \( T_g G \times T_{\alpha_g} \mathcal{G}^* \) to \( T_{\alpha_g} Z \) with the natural isomorphism from \( S \times \mathfrak{g}^* \) to \( T_g G \times T_{\alpha_g} \mathcal{G}^* \) gives a canonical isomorphism \( a_{\alpha_g} : \mathcal{G} \times \mathcal{G}^* \to T_{\alpha_g} Z \).

A direct calculation reveals that \( a_{\alpha_g}^* \Omega_{\alpha_g} \) is the symplectic form \( \Omega_{\alpha} \) on \( V = \mathfrak{g} \times \mathcal{G}^* \) defined by
\[ \Omega_{\alpha}((\xi, \phi), (\eta, \psi)) = \phi(\eta) - \psi(\xi) - \alpha[\xi, \eta] , \]
for \( \xi, \eta \in \mathfrak{g} \) and \( \phi, \psi \in \mathcal{G}^* \). It turns out that the symplectic vector spaces \( (V, \Omega_\alpha) \) and \( (V, \Omega_0) \) are canonically isomorphic. Indeed, if for \( \xi \in \mathfrak{g} \) we define \( \text{ad}'_\xi : \mathcal{G}^* \to \mathcal{G}^* \) by
\[ \alpha \in \mathcal{G}^*, \eta \in \mathfrak{g} \Rightarrow (\text{ad}'_\xi \alpha)(\eta) = \alpha[\eta, \xi] , \]
then the isomorphism \( F_{\alpha_g} : V \to V \) given by
\[ F_{\alpha_g}(\xi, \phi) = \left( \sqrt{2} \text{Ad}_{g^{-1}}' \xi, \frac{1}{\sqrt{2}} \text{Ad}_{g^{-1}}' \phi - \frac{1}{\sqrt{2}} \text{ad}'_{\text{Ad}_{g^{-1}}' \xi} \phi \right) \]
for \( \xi \in \mathfrak{g} \) and \( \phi \in \mathcal{G}^* \) satisfies
\[ F_{\alpha_g}^* \Omega_{\alpha} = \Omega_0 . \]
For \( \alpha_g \in Z \), the map \( b_{\alpha_g} = a_{\alpha_g} \circ F_{\alpha_g} \) is therefore a canonical symplectic isomorphism from \( (V, \Omega_\alpha) \) to \( (T_{\alpha_g} Z, \Omega_{\alpha_g}) \).

It is now clear that, modelling \( (Z, \Omega) \) on \( (V, \Omega_0) \), the symplectic frame bundle \( \text{Sp}(Z, \Omega) \) is canonically trivial. Indeed, \( b \) is a canonical global section. As a consequence, \( (Z, \Omega) \) has a canonical \( Mp^c \) structure: the product \( Q = Z \times Mp^c(V) \) with \( \sigma \)-equivariant bundle map \( Q \to \text{Sp}(Z, \Omega) \) determined by sending \( (z, I) \) to \( b_z \) for \( z \in Z \). If \( \pi : Q \to Z \) is the (first factor) bundle projection and \( \epsilon \) is the natural flat connexion in \( Q \to Z \), then
\[ \delta := \frac{1}{ih} \pi^* \theta + \frac{1}{2} \eta_* \epsilon \]
defines a canonical prequantum form on \( Q \). Thus, \( (Z, \Omega) \) is endowed with a distinguished prequantized \( Mp^c \) structure \( (Q, \delta) \).
We remark that the action of $G$ on $\text{Sp}(Z, \Omega)$ is given by the effect
\[ g \in G, \ z \in Z \Rightarrow g \cdot b_z = b_{g \cdot z} \circ A_g \]
on the global section $b$, where $A_g \in \text{Sp}(V)$ is given by
\[ A_g(\xi, \phi) = (Ad_g \xi, Ad_g^* \phi) \]
for $\xi \in \mathfrak{g}$ and $\phi \in \mathfrak{g}^*$.  

If we let $W \subset V$ be given by $W = \{ (x, -Ad^*_g \mu) : x \in \mathfrak{h} \}$, then $W \cap W^\perp = \mathfrak{h} \times 0$, where $\mathfrak{h}$ is the Lie algebra of $H$. We shall naturally identify the symplectic vector spaces $W/(W \cap W^\perp)$ and $\mathfrak{g}/\mathfrak{h}$; under this identification, $\nu(A_h) \in \text{Sp}(W/(W \cap W^\perp))$ corresponds to $a_h \in \text{Sp}(\mathfrak{g}/\mathfrak{h})$ given by the familiar formula
\[ \xi \in \mathfrak{g} \Rightarrow a_h(\xi + \mathfrak{h}) = Ad_h \xi + \mathfrak{h} \]
when $h \in H$.  

As a consequence of our choice of $F_{a_h}$ for $g \in G$, the global section $b$ of $\text{Sp}(Z, \Omega)$ defines over $S$ a global section of the principal $\text{Sp}(\mathfrak{g}; W)$ bundle $\text{Sp}(Z, \Omega; S)$ and so induces a global section of $\text{Sp}(E) \equiv S \times \text{Sp}(\mathfrak{g}/\mathfrak{h})$. In applying our theorems from §1, we may therefore consider the holonomy of $\delta_S$ on $H^0$-orbits in $S \times \text{Sp}(\mathfrak{g}/\mathfrak{h})$ rather than $\text{Sp}(E)$. Note that orbits of $H$ in $S \times \text{Sp}(\mathfrak{g}/\mathfrak{h})$ take the form $\{ (\beta(hg), a_hr) : h \in H \}$, for fixed $g \in G$ and $r \in \text{Sp}(\mathfrak{g}/\mathfrak{h})$.  

We make the following assertion regarding the generalized Keller-Maslov BWS rule encountered in §1.  

Claim. $\delta_S$ has trivial holonomy on $H^0$-orbits in $\text{Sp}(E) = S \times \text{Sp}(\mathfrak{g}/\mathfrak{h})$ iff there exists a homomorphism $\tau : H^0 \rightarrow Mp^c(W/(W \cap W^\perp)) = Mp^c(\mathfrak{g}/\mathfrak{h})$ such that the diagram
\[ \begin{array}{ccc}
H^0 & \xrightarrow{\tau} & Mp^c(\mathfrak{g}/\mathfrak{h}) \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
S \times \text{Sp}(\mathfrak{g}/\mathfrak{h}) & \xrightarrow{\nu} & \text{Sp}(\mathfrak{g}/\mathfrak{h})
\end{array} \]
commutes and such that
\[ (\eta \circ \tau)_* = \frac{-2}{\iota \hbar} \mu \]
on $\mathfrak{h}$.  

Proof. $(\Leftarrow)$ Let $\tau$ exist. A typical loop in an $H$-orbit in $S \times \text{Sp}(\mathfrak{g}/\mathfrak{h})$ is of the form $c_t = (p_t, a_hr)$ where $p_t = \beta(h_tg)$, for $0 \leq t \leq 1$ and fixed $g \in G$, $r \in \text{Sp}(\mathfrak{g}/\mathfrak{h})$. We claim that the $\delta_S$-horizontal lifts of $c$ are of the form $d$, given by $0 \leq t \leq 1 \Rightarrow d_t = (p_t, \tau(h_t) \tilde{r})$ for any $\tilde{r} \in Mp^c(\mathfrak{g}/\mathfrak{h})$ with $\sigma(\tilde{r}) = r$. Once this is verified, the $\delta_S$-horizontal lifts of the loop $c$ are again loops: $(\Leftarrow)$ follows. For the verification, $d$ plainly lifts $c$ since $\tau$ lifts $a$; thus we need only check $\delta_S$-horizontality, and in doing so we may assume $r = I$ and $\tilde{r} = I$ for convenience, since horizontality is right invariant. We shall use dots to denote tangent vectors to curves: thus, let $(d/dt)h_t = \dot{h}_t = (z_t)_{\text{h}}$ for $z_t \in \mathfrak{h}$. Since $\beta^* \theta = \beta$ is right invariant and $\beta_h = \mu_h$ for $h \in H$, it follows that
\[ \theta_{\beta_h}(\dot{h}_t) = \beta_h(\dot{h}_t) = \mu(z_t) ; \]
also
\[ \frac{d}{dt} (\tau(h_t)) = (\tau_* z_t)_{\tau(h_t)} . \]
Thus
\[ \delta(d_t) = \frac{1}{i\hbar}\theta_{\eta}(\hat{p}) + \frac{1}{2}\eta_*(\tau_\ast z_t) = 0 \]
by virtue of the identity \((\eta \circ \tau)_\ast = -(2/\hbar)^2 \mu\). This completes the proof of \((\Leftarrow)\).

\((\Rightarrow)\) Let \(\delta_S\) have trivial holonomy on \(H^0\)-orbits in \(S \times \text{Sp}(\mathcal{F}/\mathcal{H})\). Let \(h \in H^0\) be joined to the identity \(e\) by a path \(h_t(0 \leq t \leq 1)\) with \(h_0 = e\) and \(h_1 = h\). For any \(g \in G\) and \(r \in \text{Sp}(\mathcal{F}/\mathcal{H})\) we obtain a path
\[ 0 \leq t \leq 1 \Rightarrow c_t = (\beta(h_t g), a_r r) \]
in \(S \times \text{Sp}(\mathcal{F}/\mathcal{H})\) which we then lift \(\delta_S\)-horizontally to a path
\[ 0 \leq t \leq 1 \Rightarrow d_t = (\beta(h_t g), u_t r) \]
in \(S \times Mpc(\mathcal{F}/\mathcal{H})\) with \(u_0 = I\), for any choice of \(\hat{r}\) over \(r\). Vanishing holonomy and right invariance ensure that \(\tau(h) := u_1\) is a good definition and yields a homomorphism \(\tau: H^0 \to Mpc(\mathcal{F}/\mathcal{H})\) as claimed. \(\square\)

The following is now a direct consequence of Theorem 1 and the above claim.

**Theorem 1'.** If the homomorphism \(a: H^0 \to \text{Sp}(\mathcal{F}/\mathcal{H})\) lifts to a homomorphism \(\tau: H^0 \to Mpc(\mathcal{F}/\mathcal{H})\) satisfying \((\eta \circ \tau)_\ast = -(2/\hbar)^2 \mu\), then the canonical prequantized \(Mpc\) structure \((Q, \delta)\) for \((Z, \Omega)\) naturally confers a prequantized \(Mpc\) structure \((P', \gamma')\) on the covering \((M', \omega')\) of \(\mathcal{O}\).

Of course, in this context the lifted action of \(H^0\) on \(Q_S = S \times \text{Mpc}(\mathcal{F}/\mathcal{H})\) is given by
\[ h \cdot (x, u) = (h \cdot x, \tau(h)u) \]
for \(h \in H^0, x \in S,\) and \(u \in \text{Mpc}(\mathcal{F}/\mathcal{H})\). This makes plain the following application of Theorem 2.

**Theorem 2'.** If \(a: H \to \text{Sp}(\mathcal{F}/\mathcal{H})\) lifts to a homomorphism \(\tau: H \to Mpc(\mathcal{F}/\mathcal{H})\) with \((\eta \circ \tau)_\ast = -(2/\hbar)^2 \mu\), then the canonical prequantized \(Mpc\) structure \((Q, \delta)\) for \((Z, \Omega)\) naturally confers a prequantized \(Mpc\) structure \((P, \gamma)\) on the coadjoint orbit \(\mathcal{O} = G \cdot \mu\).

Recalling that \(\sigma_* : \text{mpc}(\mathcal{F}/\mathcal{H}) \to \text{sp}(\mathcal{F}/\mathcal{H})\) is canonically split by \(\frac{1}{i\hbar} \eta_* : \text{mpc}(\mathcal{F}/\mathcal{H}) \to u(1)\), we may reformulate the hypothesis on the existence of \(\tau\) in this theorem as follows: the Lie algebra homomorphism
\[ \left( -\frac{1}{i\hbar} \mu \right) \oplus (a_*): \mathcal{H} \to \text{mpc}(\mathcal{F}/\mathcal{H}) \]
should exponentiate to a Lie group homomorphism \(\tau: H \to Mpc(\mathcal{F}/\mathcal{H})\). In this form, the hypothesis can be compared with the familiar condition for \(\mathcal{O}\) to inherit a (Kostant-Souriau) prequantum line bundle: namely [5] that \(-(1/\hbar)^2 \mu\) should exponentiate to a unitary character \(\chi: H \to U(1)\).

**References**


Department of Mathematics, University of Florida, Gainesville, Florida 32611