CESARO AND GENERAL EULER-BOREL SUMMABILITY

LAYING TAM

(Communicated by Andrew M. Bruckner)

Abstract. The general Euler-Borel summability method is a method that includes the Euler, discrete Borel, Meyer-König, Taylor, and Karamata methods as special cases. We prove that under a certain condition the Cesàro summability of a sequence implies its summability by a general Euler-Borel method.

1. Introduction

We will give a condition under which the Cesàro summability of a sequence implies its summability by a general Euler-Borel summability method, which is also called a Sonnenschein method. The definitions of these methods are as follows.

A sequence \((s_k)\) is summable to \(S\) by the Cesàro method \(C_r\) \((r > -1)\) if

\[
\frac{r!}{n^r} s_n^r \to S \quad \text{as } n \to \infty,
\]

where

\[
s_n^r = \sum_{k=0}^{n} \binom{n - k + r - 1}{n - k} s_k.
\]

\((s_k)\) is summable to \(S\) by the general Euler-Borel method \((E, f)\), where \(f\) is a function analytic at the origin, if \((a_{n,k})\) satisfies

\[
(f(z))^n = \sum_{k=0}^{\infty} a_{n,k} z^k, \quad n = 0, 1, \ldots
\]

and

\[
\sum_{k=0}^{\infty} a_{n,k} s_k \to S \quad \text{as } n \to \infty.
\]

|z| is of course assumed to be small enough in the equation defining \(a_{n,k}\).

The Cesàro method is well known. The general Euler-Borel method has been studied in [1, 3, 6], among others. Examples of the method include the Euler method \(E_q, q > 0\), the discrete Borel method, the Meyer-König method \(S_r\),
0 < r < 1, the Taylor method $T_r$, $0 < r < 1$, and the Karamata method. The functions defining these methods are, respectively,

1. $f(z) = (z + q)/(1 + q)$.
2. $f(z) = \exp(z - 1)$.
3. $f(z) = (1 - r)/(1 - rz)$.
4. $f(z) = (1 - r)z/(1 - rz)$.
5. $f(z) = (\alpha + (1 - \alpha - \beta)z)/(1 - \beta z)$, where $\alpha < 1$, $\beta < 1$, and $\alpha + \beta > 0$.

A summability method is called regular if it sums each convergence sequence to its limit. The following theorem is due to B. Bajšanski [1].

**Theorem A.** Suppose that

1. $f$ is analytic for $|z| < R$, $R > 1$;
2. $|f(z)| < 1$ for $|z| \leq 1$, $z \neq 1$;
3. $f(1) = 1$; and
4. the number $A$ defined by

$$f(z) - z^\alpha = A|z - 1|^p + o(1)|z - 1|^p, \quad \alpha = f'(1), \quad A \neq 0, \quad \text{as } z \to 1$$

satisfies $\Re A \neq 0$.

Then the method $(E, f)$ is regular.

Condition 4 may seem to be complicated, but J. Clunie and P. Vermes have proved in [3] that if conditions 1-3 hold, then condition 4 is also necessary for the regularity of the method. The above results of Bajšanski and Clunie and Vermes were rediscovered by D. Newman [5], with a considerably shorter proof.

If $(E, f)$ satisfies the conditions of Theorem A then we will denote the number $p$ in condition 4 by $p(f)$.

It is proved in [1] that $p(f)$ is necessarily an even integer, $\Re A < 0$, and $\alpha > 0$.

All the above examples satisfy the conditions of Theorem A, with $p(f) = 2$. Moreover, in [6] it is shown that for each integer $p$ there is a method $(E, f)$ satisfying the conditions of Theorem A with $p(f) = p$.

The following example shows that in general the $C_r$ summability of a sequence does not imply its $(E, f)$ summability:

$$\sum_{k \geq 1} k^{-1/p} \exp(\beta i k^{1-1/p}), \quad \text{where } \beta > 0 \text{ and } p > 1.$$ 

This series is summable $C_r$ for every $r > 0$ but is divergent. (See [4, p. 213].) Since $\beta$ is real,

$$|k^{-1/p} \exp(\beta i k^{1-1/p})| = k^{-1/p}.$$

Since the series is divergent, the following Tauberian theorem shows that it is not $(E, f)$ summable if $p(f) = p$.

**Theorem B** [6, Corollary to Theorem 3]. If a sequence $(s_k)$ is summable by a method $(E, f)$ satisfying the conditions of Theorem A and

$$s_k - s_{k-1} = O(k^{-1/p(f)}),$$

then $(s_k)$ is convergent.

However, we have the following
Theorem. Suppose that $r$ is a positive integer. Suppose that a sequence $(s_k)$ satisfies

$$\frac{r!}{n^r} s_n^r = S + o(n^{r(1+1/p)}) \quad \text{as } n \to \infty,$$

where $p$ is an even integer. Suppose that $(E, f)$ is a method satisfying the conditions of Theorem A with $p(f) = p$. Then $(s_k)$ is summable by $(E, f)$ to $S$.

If $r = 1$, $p = 2$, and $(E, f)$ is an Euler method, then this theorem reduces to a result of K. Knopp. See [4, Theorem 149]. The special case of the theorem with $p = 2$ and $(E, f)$ equal to an Euler method, a Meyer-König method, or a Taylor method is due to D. Borwein and T. Markovich [2, Theorem 3]. They also proved that if $p = 2$ then we may replace the method $(E, f)$ by a Borel-type method or a Valiron method in the theorem. See [2, Theorems 1, 2].

2. Proof of the theorem

We will denote constants by $K$, not necessarily the same at each occurrence. Without loss of generality we may assume that $S = 0$, so that

$$s_n^r = o(n^{r/p}) \quad \text{as } n \to \infty.$$

Hence $s_n^r = e_k n^{r/p}$, where $e_k = o(1)$ as $k \to \infty$.

Let $\delta_k = \sup_{n \geq k} |e_n|$. Then $\delta_k = o(1)$ and is decreasing. Also, we have

$$|s_n^r| \leq \delta_k n^{r/p}.$$

Let $(E, f)$ be a method satisfying the conditions of Theorem A with matrix $(a_{n,k})$ and $p(f) = p$. We will divide the proof of

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{n,k} s_k = 0$$

into nine steps. The first step is similar to that of the proof of Theorem 2 in [2].

I. If $n \geq 1$ then $\sum_{k=0}^{\infty} a_{n,k} s_k = \sum_{k=0}^{\infty} (\Delta^r a_{n,k}) s_k^r$, where

$$\Delta^r a_{n,k} = \sum_{j=0}^{r} \binom{r}{j} (-1)^j a_{n,k+j}.$$

Proof. Let $m > r$. Applying Abel's partial summation formula $r$ times we have

$$\sum_{k=0}^{m} a_{n,k} s_k = \sum_{k=0}^{m-r} (\Delta^r a_{n,k}) s_k^r + \sum_{k=0}^{r-1} (\Delta^k a_{n,m-k}) s_k^{k+1}. $$

We will show that each term in the second sum tends to 0 as $m \to \infty$. Then the equality to be proved will follow.

Since $(s_k)$ is $C_r$ summable, by [4, Theorem 46], $s_k^{k+1} = o(m^r)$ for $0 \leq k < r - 1$. This is also true for $k = r - 1$, by (1) and the fact that $p > 1$. Since $n \geq 1$ and $\sum_{k=0}^{\infty} a_{n,k} z^k$ converges for some $z$ with $|z| > 1$, $a_{n,k} z^k \to 0$ as $k \to \infty$, with $|z| > 1$. Hence $(\Delta^k a_{n,m-k}) s_{m-k}^{k+1} = o(1)$ as $m \to \infty$, for $0 \leq k \leq r - 1$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
This completes the proof.

Thus to prove the theorem it suffices to prove that
\[
\sum_{k=0}^{\infty} (\Delta' a_{n,k}) s_k^r \to 0 \quad \text{as } n \to \infty.
\]

II. \( \Delta' a_{n,k} = (1/2\pi i) \int_{|z|=a} (f(z))^n z^{-k-1}z^{-r}(z-1)^r \, dz \), where \( a \) is an number in \((0, R)\), where in turn \( R \) is the number in condition 1 of Theorem A.

Proof. We have
\[
\begin{align*}
\Delta' a_{n,k} &= \sum_{j=0}^{r} \binom{r}{j} (-1)^j a_{n,k+j} \\
&= \sum_{j=0}^{r} \binom{r}{j} (-1)^j \frac{1}{2\pi i} \int_{|z|=a} (f(z))^n z^{-k-j-1} \, dz,
\end{align*}
\]
by Cauchy's integral formula,
\[
= \frac{1}{2\pi i} \int_{|z|=a} (f(z))^n z^{-k-1}z^{-r} \sum_{j=0}^{r} \binom{r}{j} (-1)^j z^{r-j} \, dz
\]
\[
= \frac{1}{2\pi i} \int_{|z|=a} (f(z))^n z^{-k-1}z^{-r}(z-1)^r \, dz.
\]
This proves II.

III. \( |\sum_{k=0}^{\infty} (\Delta' a_{n,k}) s_k^r| \leq K(T_n + U_n) \), where
\[
\begin{align*}
T_n &= \sum_{k \leq \alpha n} \delta_k k^{r/p} \int_{|z|=R_1} |f(z)|^n |z|^{-k-1}|z-1| |dz|, \\
U_n &= \sum_{k > \alpha n} \delta_k k^{r/p} \int_{|z|=R_2} |f(z)|^n |z|^{-k-1}|z-1| |dz|,
\end{align*}
\]
where \( \alpha = f'(1) \), \( R_1 = 1 - n^{-1/p} \), and \( R_2 = 1 + n^{-1/p} \). We assume that \( n \) is so large that \( R_2 < R \), the number in condition 1 of Theorem A.

Proof. In the integral in II let
\[
a = \begin{cases} 
R_1 = 1 - n^{-1/p} & \text{if } k \leq \alpha n, \\
R_2 = 1 + n^{-1/p} & \text{if } k > \alpha n.
\end{cases}
\]
Since \(|z|^{-r}\) is bounded on \(|z|=R_j, j=1, 2\), we have for \( k = 0, 1, \ldots \)
\[
|\Delta' a_{n,k}| \leq K \int_{|z|=R_j} |f(z)|^n |z|^{-k-1}|z-1| |dz|,
\]
where \( j = 1 \) if \( k \leq \alpha n \) and \( j = 2 \) if \( k > \alpha n \).

III follows from these inequalities and (2).

We note that
\[
\begin{align*}
T_n &= \sum_{k \leq \alpha n} \delta_k k^{r/p} \int_{0}^{2\pi} |f(R_1 e^{int})|^n R_1^{-k} |R_1 e^{int} - 1| \, dt, \\
U_n &= \sum_{k > \alpha n} \delta_k k^{r/p} \int_{0}^{2\pi} |f(R_2 e^{int})|^n R_2^{-k} |R_2 e^{int} - 1| \, dt.
\end{align*}
\]
IV. By condition 2 of Theorem A and given $0 < \delta < 1$, there exists $\varepsilon > 0$ and $N_\delta > 0$ such that for $n > N_\delta$ and $t \in \left[ \varepsilon, 2\pi - \varepsilon \right]$, we have

$$|f(R_j e^{it})| < 1 - \delta < 1, \quad j = 1, 2.$$  

We fix the numbers $\delta$, $\varepsilon$, and $N_\delta$.

V. $T_n = V_n + o(1)$ as $n \to \infty$, where

$$V_n = \sum_{k \leq \alpha n} \delta_k k^{r/p} \int_{-\varepsilon}^{\varepsilon} |f(R_1 e^{it})|^n R_1^{-k} |R_1 e^{it} - 1|^r \, dt.$$  

Proof. We have to prove that

$$\sum_{k \leq \alpha n} \delta_k k^{r/p} \int_{-\varepsilon}^{\varepsilon} |f(R_1 e^{it})|^n R_1^{-k} |R_1 e^{it} - 1|^r \, dt = o(1) \quad \text{as } n \to \infty.$$  

Since $\delta_k$ and $|R_1 e^{it} - 1|^r$ are bounded, if $n > N_\delta$ then by IV, the quantity on the left

$$\leq K \sum_{k \leq \alpha n} k^{r/p} \int_{-\varepsilon}^{\varepsilon} (1 - \delta)^n R_1^{-k} \, dt$$

$$\leq K (1 - \delta)^n \sum_{k \leq \alpha n} k^{r/p} (1 - n^{-1/p})^{-k}$$

$$\leq K (1 - \delta)^n n^{r/p} \sum_{k \leq \alpha n} (1 - n^{-1/p})^{-k}$$

$$\leq K (1 - \delta)^n n^{r/p} \alpha n (1 - n^{-1/p})^{-\alpha n}$$

$$= o(1) \quad \text{as } n \to \infty.$$  

Using IV again, we can prove

VI. $U_n = W_n + o(1)$ as $n \to \infty$, where

$$W_n = \sum_{k > \alpha n} \delta_k k^{r/p} \int_{-\varepsilon}^{\varepsilon} |f(R_2 e^{it})|^n R_2^{-k} |R_2 e^{it} - 1|^r \, dt.$$  

We will omit the details.

It remains to show that $V_n = o(1)$ and $W_n = o(1)$ as $n \to \infty$. To do so we need the following generalization of Lemma 1 in [1], where $r = 0$.

VII. $\int_{-\varepsilon}^{\varepsilon} |f(R_j e^{it}) R_j^{-\alpha} |n |R_j e^{it} - 1|^r \, dt = O(n^{-(r+1)/p})$ as $n \to \infty$, $j = 1$ or 2.

Proof. We will only prove that

$$\int_{0}^{\varepsilon} |f(R_1 e^{it}) R_1^{-\alpha} |n |R_1 e^{it} - 1|^r \, dt = O(n^{-(r+1)/p}) \quad \text{as } n \to \infty,$$

the rest of the proof being similar.

It follows from condition 4 of Theorem A that

$$|f(re^{it}) r^{-\alpha}| = 1 + O(1)(re^{it} - 1)^p \quad \text{as } t \to 0, \ r \to 1.$$  

Let $\psi(r, t) = \log |f(re^{it}) r^{-\alpha}|$. Then we have

$$\psi(r, t) = \log(1 + O(1)(re^{it} - 1)^p) = O(1)(re^{it} - 1)^p$$
as \( t \to 0, \ r \to 1 \). Therefore all partial derivatives of \( \psi \) of order \(< p \) are equal to 0 at \( r = 1, \ t = 0 \).

Next we estimate \((\partial^p / \partial t^p)\psi(r, t)\) in a neighborhood of \( r = 1, \ t = 0 \).

By condition 4 of Theorem A again, we have

\[
f(e^{it})e^{-iat} = 1 + A^p(e^{it} - 1)^p + o(1)(e^{it} - 1)^p \quad \text{as} \ t \to 0.
\]

Hence

\[
|f(e^{it})| = 1 + i^{2p}RA^p + o(t^p) = 1 + RA^p + o(t^p) \quad \text{as} \ t \to 0,
\]

since \( p \) is even.

Thus

\[
\psi(1, t) = \log|f(e^{it})| = \log(1 + RA^p + o(t^p)) = RA^p + o(t^p) \quad \text{as} \ t \to 0.
\]

This implies that \((\partial^p / \partial t^p)\psi(1, 0) = (p!)RA\). Since \( RA < 0 \), if \( r \) is close to 1 and \(|t| < \varepsilon\) then we have

\[
\frac{1}{p!} \frac{\partial^p}{\partial t^p} \psi(r, t) < -M
\]

for some positive constant \( M \). (We may need to decrease \( \varepsilon \), which we have fixed in step IV, but this does no harm.)

By Taylor’s formula,

\[
(3) \quad \psi(r, t) = \sum_{m=0}^{p-1} C_m(r)t^m + C_p(r, t)t^p,
\]

where

\[
C_m(r) = \frac{1}{m!} \frac{\partial^m \psi(r, t)}{\partial t^m} \bigg|_{t=0} \quad \text{for} \ m = 0, 1, \ldots, p-1,
\]

\[
C_p(r, t) = \frac{1}{p!} \frac{\partial^p \psi(r, t)}{\partial t^p} \bigg|_{t=\tau}, \quad |\tau| < |t| < \varepsilon.
\]

Thus \( C_p(r, t) < -M \) if \( r \) is close to 1 and \(|t| < \varepsilon\).

Since all partial derivatives of \( \psi \) of order \(< p \) vanish at \( r = 1, \ t = 0 \), and since

\[
\frac{d^n C_m(r)}{dr^n} = \frac{1}{m!} \frac{\partial^{n+m} \psi(r, t)}{\partial r^n \partial t^m} \bigg|_{t=0},
\]

all derivatives of \( C_m(r) \) of order \(< p - m \) vanish at \( r = 1 \). Hence \( r = 1 \) is a zero of order at least \( p - m \) of \( C_m(r) \) and there exist constants \( K_m > 0 \) such that

\[
C_m(r) \leq K_m|r - 1|^{p-m}
\]

for \( m = 0, 1, \ldots, p-1 \), if \( r \) is close to 1. It now follows from (3) and the estimates of \( C_p(r, t) \) and \( C_m(r) \) that if \(|t| < \varepsilon\) and \( n \) is large enough, so that \( R_1 = 1 + n^{-1/p} \) is close enough to 1, then

\[
\psi(R_1, t) \leq \sum_{m=0}^{p-1} K_m|R_1 - 1|^{p-m}t^m - Mt^p
\]

\[
\leq \sum_{m=0}^{p-1} K_m n^{-1+m/p}t^m - Mt^p.
\]
On the other hand, an easy computation shows that
\[ |R_1 e^{it} - 1|^{r} \leq n^{-r/p}(1 + 2n^{2/p}(1 - \cos t))^{r/2}. \]

Finally, we have
\[
\int_{0}^{\varepsilon} |f(R_1 e^{it})| R_1^{-\alpha}|n| R_1 e^{it} - 1|^{r} dt
\]
\[
= \int_{0}^{\varepsilon} \{\exp(n\psi(R_1, t))\} |R_1 e^{it} - 1|^{r} dt
\]
\[
\leq n^{-r/p} \int_{0}^{\varepsilon} \left\{ \exp \left( n \left( \sum_{m=0}^{p-1} K_m n^{1+m/p} t^m - M t^p \right) \right) \right\} (1 + 2n^{2/p}(1 - \cos t))^{r/2} dt
\]
\[
\leq n^{-(r+1)/p} \int_{0}^{\varepsilon} \left\{ \exp \left( \sum_{m=0}^{p-1} K_m n^m - M n^p \right) \right\} (1 + 2n^{2/p}(1 - \cos n^{-1/p} v))^{r/2} dv
\]
(We have made the substitution \( v = n^{1/p} t \).)
\[
= O(n^{-(r+1)/p}) \text{ as } n \to \infty.
\]

This completes the proof of VII.

VIII. \( V_n = o(1) \text{ as } n \to \infty \).

Proof. We have
\[
V_n = \sum_{k \leq \alpha n} \delta_k k^{r/p} \int_{-\varepsilon}^{\varepsilon} |f(R_1 e^{it})|^n R_1^{-k}|R_1 e^{it} - 1|^{r} dt
\]
\[
= \sum_{k \leq \alpha n} \delta_k k^{r/p} R_1^{\alpha n-k} \int_{-\varepsilon}^{\varepsilon} |f(R_1 e^{it})| R_1^{-\alpha}|n| R_1 e^{it} - 1|^{r} dt
\]
\[
\leq K \sum_{k \leq \alpha n} \delta_k k^{r/p} R_1^{\alpha n-k} n^{-(r+1)/p}, \text{ by VII},
\]
\[
\leq K n^{-(r+1)/p} \sum_{k \leq \alpha n} \delta_k k^{r/p} (1 - n^{-1/p})^{\alpha n-k},
\]
since \( R_1 = 1 - n^{-1/p} \).

Let \( c > 0 \). Since \( \delta_k \) tends to 0, there exists \( N_c > 0 \) such that if \( k > N_c \), then \( \delta_k < c \). Hence if \( \alpha n > N_c \), then
\[
V_n \leq K n^{-(r+1)/p} (1 - n^{-1/p})^{\alpha n} \sum_{k \leq N_c} \delta_k k^{r/p} (1 - n^{-1/p})^{-k}
\]
\[
+ K n^{-(r+1)/p} \sum_{N_c < k \leq \alpha n} c k^{r/p} (1 - n^{-1/p})^{\alpha n-k}.
\]

For each fixed \( c > 0 \), the first term on the right
\[
= K n^{-(r+1)/p} (1 - n^{-1/p})^{\alpha n} O(1) = o(1) \text{ as } n \to \infty .
\]
The second term

\[ \leq K \alpha^{-\frac{m}{1-p}} \sum_{k \geq \alpha} k^{r/p} R_k^{\alpha - k} \]

\[ \leq K \alpha^{-\frac{m}{1-p}} \sum_{m=0}^{\infty} (1 - \frac{1}{1-\frac{1}{n^{1/p}}})^m \]

\[ \leq K \alpha^{-\frac{m}{1-p}} \frac{1}{1 - (1 - \frac{1}{n^{1/p}})} = Kc. \]

Since \( c \) is an arbitrary positive number, \( V_n = o(1) \) as \( n \to \infty \).

**IX.** \( W_n = o(1) \) as \( n \to \infty \).

**Proof.** Since \( \delta_k \) is decreasing,

\[ W_n \leq \delta_{[\alpha n]} \sum_{k \geq \alpha} k^{r/p} R_k^{\alpha - k} \int_{-\epsilon}^{\epsilon} |f(R_2 e^{it}) R_2^{-\alpha n} | R_2 e^{it} - 1 |^r \, dt \]

\[ \leq K \delta_{[\alpha n]} n^{-\frac{m}{1-p}} \sum_{k \geq \alpha} k^{r/p} R_k^{\alpha - k}, \quad \text{by VII.} \]

Let \( m = [\alpha n] \) and \( R = R_2^{-1} \). Then we have

\[ W_n \leq K \delta_{[\alpha n]} n^{-\frac{m}{1-p}} R_2^{\alpha n} \sum_{k \geq m} k^{r/p} R_k. \]

We will prove that

\[ n^{-\frac{m}{1-p}} R_2^{\alpha n} \sum_{k \geq m} k^{r/p} R_k = O(1) \]

and \( n \to \infty \). Then since \( \delta_n = o(1) \), IX follows.

Let \( q \) be the smallest integer satisfying \( q \geq r/p \). Then the sum in (4)

\[ \leq K n^{r/p-q} \sum_{k \geq m} k^q R_k \]

\[ \leq K n^{r/p-q} \sum_{k \geq m} (k+1)(k+2)\cdots(k+q)R_k \]

\[ = K n^{r/p-q} \frac{d^q}{dR^q} \sum_{k \geq m} R^{k+q} \]

\[ = K n^{r/p-q} \frac{d^q}{dR^q} \frac{R^{m+q}}{1-R} \]

\[ = K n^{r/p-q} \frac{\sum_{j=0}^{q} \binom{q}{j} (m+q)(m+q-1)\cdots(m+j+1)R^{m+j} j!}{(1-R)^{j+1}}. \]

Since

\[ (m+q)(m+q-1)\cdots(m+j+1) \leq Kn^{q-j}, \]
and

\[
\frac{1}{(1 - R)^{j+1}} \leq Kn^{(j+1)/p},
\]

the sum in (4)

\[
\leq Kn^{r/p-q} \sum_{j=0}^{q} \binom{q}{j} n^{q-j} R^{m+j} n^{(j+1)/p} \leq KR^m n^{(r+1)/p}.
\]

Since \( R = R^{-1} \), this implies that the quantity on the left of (4) is bounded. This proves IX.

The proof of the theorem is complete.

ACKNOWLEDGMENT

I would like to thank Professor Bogdan Baishanski for many helpful discussions and for simplifying the proof of IX. I would also like to thank the referee for his comments.

REFERENCES


DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, 231 W. 18TH AVENUE, COLUMBUS, OHIO 43210
Current address: Flat 1625, 16 Lai Tak Tsuen Road, Tsuen Wing Lau, Hong Kong