CURVATURE PINCHING FOR THREE-DIMENSIONAL MINIMAL
SUBMANIFOLDS IN A SPHERE

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Abstract. In this paper, some pinching theorems for the Ricci curvature and
the scalar curvature of three-dimensional compact minimal submanifolds in a
sphere are given.

1. Introduction

Let $M^n$ be an $n$-dimensional compact orientable minimal submanifold in
a unit $(n + p)$-sphere $S^{n+p}$. In [2] it was proved that if $n \geq 4$ and the Ricci
curvature of $M^n$ is larger than $n - 2$, then $M^n$ is totally geodesic in $S^{n+p}$. 
Recently, the corresponding problem for the three-dimensional case was treated
in [4]. The aim of this paper is to improve the result of [4] so that the theorem
of [2] is valid for the case $n = 3$. Precisely, we prove

Theorem 1. Let $M^3$ be a three-dimensional compact minimal submanifold in
a unit sphere $S^{3+p}$. If the Ricci curvature of $M^3$ is larger than 1 then $M^3$
is totally geodesic in $S^{3+p}$.

Moreover, for lower codimension, we have

Theorem 2. Let $M^3$ be a compact orientable minimal submanifold in $S^{3+p}$
with $p \leq 2$. If the Ricci curvature of $M^3$ is not less than $(5p - 4)/(4p - 2)$ then $M^3$
is totally geodesic in $S^{3+p}$.

In the same way as in the proof of Theorem 1, we also obtain

Theorem 3. Let $M^3$ be a compact minimal submanifold in $S^{3+p}$. If the scalar
curvature of $M^3$ is larger than 4 then $M^3$ is totally geodesic.

Throughout this paper, all the manifolds dealt with are smooth and con-

nected.

2. Preliminaries

In this section we state some notations and basic formulas. More details can
be found in [4]. Let $M^3$ be a three-dimensional compact Riemannian manifold

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that is minimally immersed in a unit \((3+p)\)-sphere \(S^{3+p}\). We choose a local field of orthonormal frames \(e_1, \ldots, e_{3+p}\) in \(S^{3+p}\) such that, restricted to \(M^3\), the vectors \(e_1\), \(e_2\), and \(e_3\) are tangent to \(M^3\). Unless otherwise stated, we agree on the following ranges of indices: \(1 \leq i, j, k, \ldots \leq 3; 4 \leq \alpha, \beta, \ldots \leq 3+p\). The second fundamental form of \(M^3\) in \(S^{3+p}\) is

\[
\sigma = \sum_{\alpha, i, j} h_{ij}^{\alpha} \omega^i \otimes \omega^j \otimes e_{\alpha},
\]

of which the length square is \(\|\sigma\|^2 = \sum_{\alpha, i, j} (h_{ij}^{\alpha})^2\).

Let \(UM \to M^3\) be the unit tangent bundle over \(M^3\). We define a function \(f: UM \to \mathbb{R}\) by

\[
f(u) = \|\sigma(u, u)\|^2 = \sum_{\alpha} \left( \sum_{i, j} h_{ij}^{\alpha} u^i u^j \right)^2
\]

for \(u = \sum_i u_i e_i \in UM\). Since \(UM\) is compact, \(f\) attains its maximum at a vector in \(UM\). Suppose that this vector is \(v \in UM_{x_0}\) for some point \(x_0 \in M^3\). By taking \(e_1 = v\) at \(x_0\) and letting

\[
b_{ij} = \sum_{\alpha} h_{i1}^{\alpha} h_{1j}^{\alpha},
\]

from the maximality of \(f\) we can choose vectors \(e_2\) and \(e_3\) at \(x_0\) such that (cf. [4])

\[
f(v) = b_{11} = \max_{u \in UM} \{ \|\sigma(u, u)\|^2 \},
\]

\[
b_{ij} = 0 \quad (i \neq j),
\]

\[
2 \sum_{\alpha} (h_{1k}^{\alpha})^2 + b_{kk} - b_{11} \leq 0 \quad (k \neq 1),
\]

\[
\sum_{\alpha} (h_{11i}^{\alpha})^2 + \sum_{\alpha} h_{i1}^{\alpha} h_{11i}^{\alpha} \leq 0
\]

at the point \(x_0\).

The Gauss equation of \(M^3\) is

\[
R_{ijkl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} + \sum_{\alpha} (h_{ik}^{\alpha} h_{jl}^{\alpha} - h_{il}^{\alpha} h_{jk}^{\alpha}),
\]

from which and the minimality it follows that

\[
R_{ij} = 2\delta_{ij} - \sum_{\alpha, k} h_{ik}^{\alpha} h_{jk}^{\alpha}
\]

and

\[
R = 6 - \|\sigma\|^2,
\]

where \(R_{ijkl}\), \(R_{ij}\), and \(R\) denote the curvature tensor, the Ricci tensor, and the scalar curvature of \(M^3\), respectively.

Summing up for \(i\) in (2.7) and using (2.5) and the Ricci identity, we easily get [4]

\[
0 \geq 3f(v) + 2 \sum_{\alpha, k \neq 1} b_{kk} (h_{1k}^{\alpha})^2 - 2f(v) \sum_{\alpha, k \neq 1} (h_{1k}^{\alpha})^2 - \sum_{k \neq 1} (b_{kk})^2 - f(v) b_{11}
\]

at the point \(x_0\).
Finally, as is well known, the curvature tensor of a three-dimensional manifold can be expressed as
\begin{equation}
R_{ijkl} = \delta_{ik} R_{jl} - \delta_{il} R_{jk} + \delta_{jl} R_{ik} - \delta_{jk} R_{il} - \frac{1}{2} R (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}).
\end{equation}

3. Proofs of Theorems 1 and 3

We restrict ourselves to the point $x_0$ where the function $f$ defined by (2.2) attains its maximum. Then, from (2.3) and (2.4) one can easily see that
\begin{equation}
(b_{kk})^2 \leq \left( \sum_a (h_{11}^a)^2 \right) \left( \sum_a (h_{kk}^a)^2 \right) \leq (b_{11})^2
\end{equation}
for $k \neq 1$, from which and the three-dimensional minimality it follows that
\begin{equation}
b_{22} \leq 0, \quad b_{33} \leq 0, \quad \sum_{k \neq 1} (b_{kk})^2 \leq \left( \sum_{k \neq 1} b_{kk} \right)^2 = (b_{11})^2.
\end{equation}

From (2.3) and (2.9) we have
\begin{equation}
- \sum_{a, k \neq 1} (h_{ik}^a)^2 = R_{11} - 2 + b_{11}.
\end{equation}

Substituting (3.3) into (2.11) and using (3.2), one can obtain
\begin{equation}
0 \geq 3 f(v) + 2 \sum_{a, k \neq 1} b_{kk} (h_{ik}^a)^2 + 2 f(v) (R_{11} - 2 + b_{11}) - \sum_{k \neq 1} (b_{kk})^2 - f(v) b_{11}
\end{equation}
\begin{equation}
= -f(v) + 2 \sum_{a, k \neq 1} b_{kk} (h_{ik}^a)^2 + 2 f(v) R_{11} + f(v) b_{11} - \sum_{k \neq 1} (b_{kk})^2
\end{equation}
\begin{equation}
\geq -f(v) + 2 \sum_{a, k \neq 1} b_{kk} (h_{ik}^a)^2 + 2 f(v) R_{11}.
\end{equation}

On the other hand, by (3.2), (2.6), and (3.1), we have respectively
\begin{equation}
\sum_{a, k \neq 1} b_{kk} (h_{ik}^a)^2 \geq \frac{1}{2} \sum_{k \neq 1} b_{kk} (b_{11} - b_{kk}) = -\frac{1}{2} \sum_i (b_{ii})^2
\end{equation}
and
\begin{equation}
\sum_{a, k \neq 1} b_{kk} (h_{ik}^a)^2 \geq -f(v) \sum_{a, k \neq 1} (h_{ik}^a)^2 = f(v) (R_{11} - 2 + b_{11}).
\end{equation}

Introducing (3.5) and (3.6) into (3.4), we get
\begin{equation}
0 \geq -f(v) + 2 f(v) R_{11} + f(v) (R_{11} - 2) + \frac{1}{2} \left[ (b_{11})^2 - \sum_{k \neq 1} (b_{kk})^2 \right]
\geq 3 f(v) (R_{11} - 1).
\end{equation}

Thus, if the Ricci curvature of $M^3$ is larger than 1 then (3.7) implies that $f(v) = 0$, i.e., $\|\sigma\|^2$ vanishes identically. This proves Theorem 1.
In the similar manner, it follows from (2.11), (3.5), (3.6), and (3.1) that

\[ 0 \geq 3f(v) + \left( \sum_{\alpha, k \neq 1} b_{kk}(h_{1k}^\alpha)^2 - 2f(v) \sum_{\alpha, k \neq 1} (h_{1k}^\alpha)^2 \right) \]

\[ \geq 3f(v) - 3f(v) \sum_{\alpha, k \neq 1} (h_{2k}^\alpha)^2 - \frac{3}{2} f(v)b_{11} - \frac{3}{2} \sum_{k \neq 1} (b_{kk})^2 \]

\[ \geq \frac{3}{2} f(v) \left\{ 2 - \sum_{\alpha, i} (h_{ii}^\alpha)^2 - 2 \sum_{\alpha, k \neq 1} (h_{2k}^\alpha)^2 \right\} \]

\[ \geq \frac{3}{2} f(v) \{ 2 - \|\sigma\|^2(x_0) \}. \]

Thus, if the scalar curvature of \( M^3 \) is larger than 4, i.e., \( \|\sigma\|^2 < 2 \), then (3.8) implies that \( f(v) = 0 \), i.e., \( M^3 \) is totally geodesic. Theorem 3 is proved.

4. Proof of Theorem 2

For a compact orientable minimal submanifold \( M^3 \) in \( S^{3+p} \), a standard calculation gives (cf. [4, Lemma 1.2])

\[ \int_{M^3} \left\{ 2 - \sum_{\alpha, i, j, k, l} h_{ij}^\alpha (h_{kli}^\alpha R_{ljk} + h_{kli}^\alpha R_{lkj}) + \frac{1}{p}\|\sigma\|^4 - 3\|\sigma\|^2 \right\}^* 1 \leq 0, \]

where \( *1 \) denotes the volume element of \( M^3 \).

Let \( Q(x) \) be the function assigns to each point \( x \) of \( M^3 \) the minimum of the Ricci curvatures of \( M^3 \) at that point \( x \). For each \( \alpha \), let \( \alpha_i \) be the eigenvalues of the matrix \( (h_{ij}^\alpha) \). Then, by (2.12) we have

\[ \sum_{i,j,k,l} h_{ij}^\alpha (h_{kli}^\alpha R_{ljk} + h_{kli}^\alpha R_{lkj}) = \sum_{i \neq j} (\alpha_i^2 - \alpha_i \alpha_j)(R_{ii} + R_{jj} - \frac{1}{2}R) \]

\[ = 3 \sum_i \alpha_i^2 R_{ii} - \frac{1}{2} R \sum_i \alpha_i^2 \geq (3Q - \frac{1}{2}R) \sum_{i,j} (h_{ij}^\alpha)^2, \]

from which and (4.1) it follows that

\[ \int_{M^3} \|\sigma\|^2(6Q - R + \frac{1}{p}\|\sigma\|^2 - 3)^*1 \leq 0, \]

i.e., by (2.10),

\[ \int_{M^3} \|\sigma\|^2 \left( 6Q - 9 + \frac{p+1}{p}\|\sigma\|^2 \right)^*1 \leq 0. \]

On the other hand, the well-known Simons inequality [5] for \( n = 3 \) is

\[ \int_{M^3} \|\sigma\|^2 \left( \frac{3p}{2p-1} - \|\sigma\|^2 \right)^*1 \leq - \int_{M^3} \|\nabla \sigma\|^2 *1 \leq 0, \]
from which and (4.2) we get

\[(4.4) \quad \int_{M^3} \|\sigma\|^2 \left( Q - \frac{5p - 4}{4p - 2} \right)^* \leq 0. \]

Thus, if \( Q > \frac{(5p - 4)/(4p - 2)} \), then (4.4) implies that \( \|\sigma\|^2 = 0 \) identically.

We now consider the case that \( Q = \frac{(5p - 4)/(4p - 2)} \). Then, (4.2) becomes

\[\int_{M^3} \|\sigma\|^2 \left( \|\sigma\|^2 - \frac{3p}{2p - 1} \right)^* \leq 0,\]

which together with (4.3) gives that \( \nabla \sigma = 0 \), and hence, since \( \|\sigma\|^2 \) is constant, \( \|\sigma\|^2 = 0 \) or \( 3p/(2p - 1) \). Since the Ricci curvature of \( M^3 \) is positive everywhere, \( M^3 \) cannot be the Clifford hypersurface. Now, Theorem 2 follows directly from the well-known result of [1] for \( n = 3 \).

Remark. It is clear that the pinching values given here are not the best possible. In general, for each pair \((n, p)\), there is a best pinching value for minimal \( M^n \) in \( S^{n+p} \). Really, in [2] the pinching constant \( n - 2 \) for the Ricci curvature is not sharp for \( n \neq 4 \) and \( p \neq 1 \). In [3], it was proved that there exists an isometric minimal immersion of \( S_3^{1/8} \) into \( S^9 \), where \( S_3^{1/8} \) denotes the 3-sphere with constant sectional curvature \( 1/8 \). On the other hand, it is well known that every three-dimensional Einstein manifold is of constant curvature. So, perhaps one can surmise that the best possible pinching value of the Ricci curvature for minimal \( M^3 \) in \( S^{3+p} \) would be \( \frac{1}{4} \). However, we have not demonstrated it.

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